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SPECTRAL THEORY OF SMITH GRAPHS

DRAGOŠ CVETKOVIĆ

Dedicated to Professor Ivan Gutman

(Presented at the 4th Meeting, held on May 25, 2017)

A b s t r a c t. Graphs whose spectrum belongs to the interval $[-2, 2]$ *are called Smith graphs. The structure of a Smith graph with a given spectrum depends on a system of Diofantine linear algebraic equations. We establish several properties of this system and show how it can be simplified and effectively applied.*

AMS Mathematics Subject Classification (2010): 05C50.

Key Words: spectral graph theory, spectral radius, Diophantine equations.

1. *Introduction*

Let G be a simple graph on n vertices (or of order n), and adjacency matrix A. The characteristic polynomial of A (equal to $\det(xI - A)$) is also called the *characteristic polynomial* of G. The eigenvalues and the spectrum of A (which consists of n eigenvalues) are called the *eigenvalues* and the *spectrum* of G, respectively. Since A is real and symmetric, its eigenvalues are real. The eigenvalues of G (in non-increasing order) are denoted by $\lambda_1, \ldots, \lambda_n$. In particular, λ_1, as the largest eigenvalue of G, will be called the *spectral radius* (or *index*) of G.

The spectrum of G (as a multiset or family of reals) will be denoted by $\widehat{G}$. The disjoint union of graphs G_1 and G_2 will be denoted by $G_1 + G_2$, while the union of their spectra (i.e. the spectrum of $G_1 + G_2$) will be denoted by $\widehat{G}_1 + \widehat{G}_2$; in addition, kG ($k\widehat{G}$) stands for the union of k copies of G (resp. $\widehat{G}$).

We shall use a more general setting from [5].

A mapping ϕ from a finite set S to the integer set $\mathbb{Z}$ is called a *family (system)* over S (as an underlying set). For $x \in S$ the value $\phi(x)$ is the *multiplicity* of x in the family ϕ. This definition extends the notion of an ordinary family; normally we would allow only non-negative multiplicities of elements in ordinary families, while here multiplicities could be negative.

Let $\mathbf{X}, \mathbf{Y}$ be families of elements of a set S. For $k \in \mathbb{Z}$ we define $k\mathbf{X}$ to be the family obtained from $\mathbf{X}$ by multiplying the multiplicities of its elements by k. The *union* $\mathbf{X} + \mathbf{Y}$ of families $\mathbf{X}, \mathbf{Y}$ is the family consisting of elements contained in any of the two families with multiplicities being the sums of multiplicities in the corresponding families. The family $k_1\mathbf{X_1} + \cdots + k_n\mathbf{X_n}$ ($k_1, \ldots, k_n \in \mathbb{Z}$) is called a *linear combination* of families $\mathbf{X_1}, \ldots, \mathbf{X_n}$.

The set of all families over a set S is an Abelian group with respect to the union + of families and also a $\mathbb{Z}$-module. It can be interpreted as a set of integral vectors of dimension $|S|$ wth usual adition and multiplication by a scalar.

The corresponding "subtraction" operation $-$ is introduced in a standard manner and used in treating graph spectra in [3].

The problem of determining the graphs by spectral means is one of the oldest problems in the spectral graph theory. This problem is studied in the literature for various kinds of graph spectra (based on different types of graph matrices). Here we have in mind the adjacency matrix.

We say that two (non-isomorphic) graphs are *cospectral* if their spectra coincide. On the other hand, we say that a graph is determined by its spectrum if it is a unique graph having this spectrum. As in [8], we use DS (non-DS) to indicate that some graph is determined (resp. non-determined) by its spectrum. Many results on spectral characterizations can be found in [8]. For early results see [2].

The *cospectral equivalence class* of a graph G is the set of all graphs cospectral to G (including G itself).

We consider the class of graphs whose spectral radius is at most 2. This class includes, for example, the graphs whose each component is either a path or a cycle.

All graphs with the spectral radius at most 2 have been constructed by J.H. Smith [11]. Therefore these graphs are usually called the *Smith graphs*. Eigenvalues of these graphs have been determined in [3]. All eigenvalues are of the form $2\cos\frac{p}{q}\pi$, where p, q are integers and $q \neq 0$. For a review of [3] by J.H. Smith see [6], pp. 78–79, claiming also that the form of eigenvalues of Smith graphs follows from an old theorem by L. Kronecker [9].

A path (cycle) on n vertices will be denoted by P_n (resp. C_n).

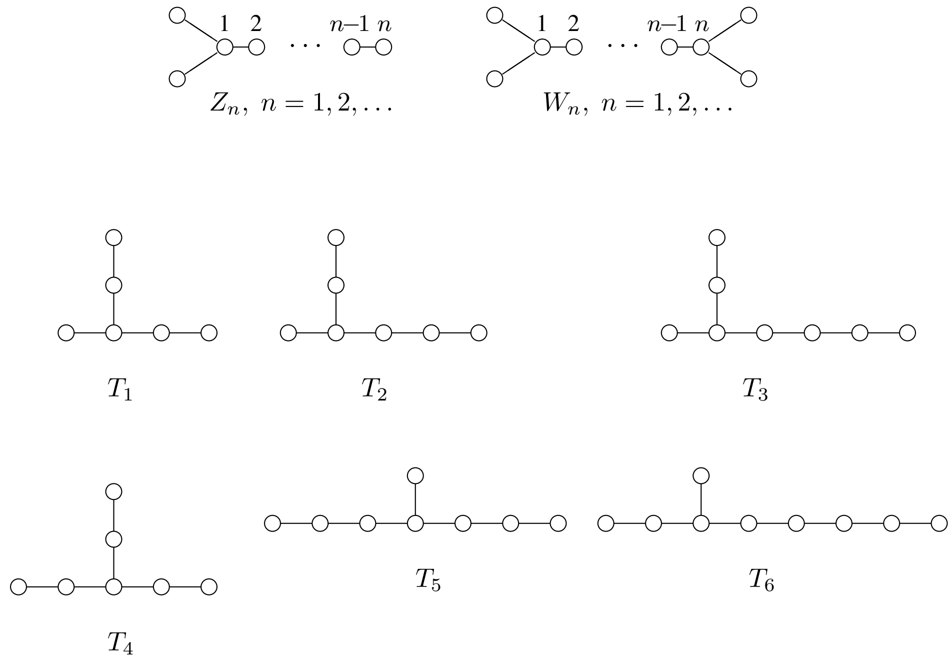

Figure 1. Some of the Smith graphs

A connected graph with index ≤ 2 is either a cycle C_n $(n = 3, 4, \ldots)$, or a path P_n $(n = 1, 2, \ldots)$, or one of the graphs depicted in Fig. 1 (see [11]). Note that W_1 coincide with the star $K_{1,4}$, while Z_1 with P_3. In addition, the graphs C_n, W_n, T_4, T_5, and T_6 are connected graphs with index equal to 2; all other graphs, namely, P_n, Z_n, T_1, T_2 and T_3 are the induced subgraphs of these graphs (so the index of each of them is less than 2). The graph Z_n is called a *snake* while W_n is a *double snake*. The trees T_1, T_2, T_3, T_4, T_5, and T_6 will be called *exceptional Smith graphs*. We denote the set of all these graphs by $\mathcal{S}^*$; the set of those which are bipartite, so odd cycles are excluded, will be denoted by $\mathcal{S}$. The spectrum of each graph from $\mathcal{S}^*$ can be found (in an explicit form) in [3].

Spectra of Smith graphs are related to Coxeter groups and Coxeter systems (see, for example, [1], p. 84 and p. 294).

Let G be any graph whose each component belongs to $\mathcal{S}^*$, we can write

$$G = \sum_{S \in \mathcal{S}^*} r(S)S, \tag{1.1}$$

where $r(S) \geq 0$ is a repetition factor (tells how many times S is appearing as a component in G).

The repetition factor $r(S_i)$ of some of the graph $S_i \in \mathcal{S}^*$ for any relevant index i will be denoted by s_i. So we have non-negative integers

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, t_1, t_2, t_3, t_4, t_5, t_6.$$

We have omitted z_1 since $Z_1 = P_3$ and the variable p_3 is relevant. We shall use $c_2, c_3, \ldots,$ for repetition factors of the even cycles $C_4, C_6, \ldots$.

For non-bipartite graphs from $\mathcal{S}^*$ we have to introduce variables $o_3, o_5, o_7, \ldots$ counting the numbers of odd cycles $C_3, C_5, C_7, \ldots$.

For a given graph $G \in \mathcal{S}^*$ the above variables which do not vanish, together with their values, are called *parameters* of G. Parameters of a graph indicate the actual number of components of particular types present in G.

The paper [3] has given foundations of spectral theory of Smith graphs. It has much content but relies much on the intuition of the reader. In this paper we shall add many relevant details and explain some particular topics.

This paper can be considered as a continuation of the research initiated in [3] and further extended in [7] and [4]. See also [4] for references to some related results on spectral determination of Smith graphs.

Among other things, in [3] an effective procedure which enables the determination of all graphs having the spectrum equal to a given system of numbers of the form $2\cos\frac{p}{q}\pi$ is exposed. These graphs can be obtained by solving a system of linear Diofantine equations. The importance of this procedure was explained in [2], p. 189. Namely, in general case, given a hypothetical spectrum, we do not know how to decide whether a graph with this spectrum exists apart from considering all graphs with the corresponding number of vertices.

The rest of the paper is organized as follows. Section 2 contains some preliminary results including the description of a system of linear Diophantine equations for parameters of a Smith graph. Section 3 describes an algorithmic criterion for cospectrality of Smith graphs. Section 4 contains several general results on the system of equations. In Section 5 we consider our system in some detail. In Section 6 we show how the system can be reduced and simplified. Section 7 gives a survey of special cases of the system. Appendix contains some remarks on calculation of spectra of Smith graphs.

2. *Preliminary results*

Spectra of connected Smith graphs have been determined in [3] and they are reproduced in Appendix. On the basis of the determined spectra the following equalities have been obtained in [3]:

$$
\begin{aligned}
\widehat{W}_n &= \widehat{C}_4 + \widehat{P}_n, \\
\widehat{Z}_n + \widehat{P}_n &= \widehat{P}_{2n+1} + \widehat{P}_1, \\
\widehat{C}_{2n} + 2\widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_{n-1}, \\
\widehat{T}_1 + \widehat{P}_5 + \widehat{P}_3 &= \widehat{P}_{11} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_2 + \widehat{P}_8 + \widehat{P}_5 &= \widehat{P}_{17} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_3 + \widehat{P}_{14} + \widehat{P}_9 + \widehat{P}_5 &= \widehat{P}_{29} + \widehat{P}_4 + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_4 + \widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_2, \\
\widehat{T}_5 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_3 + \widehat{P}_2, \\
\widehat{T}_6 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_4 + \widehat{P}_2.
\end{aligned} \tag{2.1}
$$

The following theorem is taken from [3]. Note that it deals only with bipartite graphs from $\mathcal{S}^*$.

Theorem 2.1. *Let $H \in \mathcal{S}$. Then the spectrum $\widehat{H}$ of H has the following representation*

$$\widehat{H} = \sigma_0 \widehat{C}_4 + \sum_{i=1}^{m} \sigma_i \widehat{P}_i,$$

where $\sigma_0 \geq 0$, $m \geq 0$ and $\sigma_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$ and $\sigma_m > 0$ (if $m > 0$). Moreover, this representation is unique.

In the sequel, the representation of the spectrum of $G \in \mathcal{S}$ given by Theorem 2.1 will be called *canonical*. The integers σ_0 and σ_i for $1 \leq i \leq m$ represent the *coefficients* of such representation. This representation for all bipartite graphs from $\mathcal{S}$ can be obtained by using the equalities (2.1). Parameter m is called the *height* of the representation.

Non-bipartite Smith graphs contain odd cycles as components. It was described in [3] how these odd cycles can be identified. They can be deleted fron the graph and their eigenvalues deleted from the spectrum. The remaining graph is bipartite. Hence, we may assume that considered graphs are bipartite and we shall do so in this paper without the loss of generality.

Next we shall describe the procedure from [3] for constructing all Smith graphs with given spectrum.

Let consider only bipartite graphs. As it is well known, bipartite graphs have a symmetric spectrum with respect to the zero point. Given a symmetric system (family) L of numbers of the form $2\,\cos\frac{p}{q}\pi$, we try to represent it as s linear combination of $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$. If this is not possible, L is not a spectrum of any graph (according to Theorem 2.1). In the case such a representation is possible, the mentioned linear combination is unique. Principles of finding the corresponding coefficients are clear since among $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$ no two systems have the same greatest element. Some details will be given in Section 3.

Let now L be represented as:

$$L = \sigma_0 \widehat{C}_4 + \sigma_1 \widehat{P}_1 + \sigma_2 \widehat{P}_2 + \cdots + \sigma_m \widehat{P}_m. \tag{2.2}$$

Suppose that L is the spectrum of a graph G. Presenting L as s linear combination of spectra of the components we get:

$$\begin{aligned} L =& p_1\widehat{P}_1 + p_2\widehat{P}_2 + p_3\widehat{P}_3 + \cdots + z_2\widehat{Z}_2 + z_3\widehat{Z}_3 + \cdots \\ &+ w_1\widehat{W}_1 + w_2\widehat{W}_2 + w_3\widehat{W}_3 + \cdots + c_2\widehat{C}_4 + c_3\widehat{C}_6 + \cdots \\ &+ t_1\widehat{T}_1 + t_2\widehat{T}_2 + t_3\widehat{T}_3 + t_4\widehat{T}_4 + t_5\widehat{T}_5 + t_6\widehat{T}_6, \end{aligned} \tag{2.3}$$

for some non-negative integers (parameters of G)

$$\begin{aligned} &p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, \\ &c_2, c_3, \ldots, t_1, t_2, t_3, t_4, t_5, t_6. \end{aligned} \tag{2.4}$$

The number of terms in (2.3), as well as in (2.5) – (2.8) is finite. In each particular case actual terms should be identified (see examples in Sections 4 – 7).

Using the relations (2.1) one can express the equation (2.3) in the form:

$$L = F_0\widehat{C}_4 + F_1\widehat{P}_1 + F_2\widehat{P}_2 + \cdots, \tag{2.5}$$

where the coefficients F_i $i = 0, 1, \ldots$ in (2.5) are functions of variables (2.4). Hence,

$$F_0 = (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots) + t_4 + t_5 + t_6, \tag{2.6}$$

$$F_1 = p_1 + w_1 + (z_2 + z_3 + \cdots) - 2\,(c_3 + c_4 + \cdots) + t_1 + t_2 + t_3 - t_4 - t_5 - t_6, \tag{2.7}$$

and for $i > 1$ and $i \neq 2, 3, 4, 5, 8, 9, 11, 14, 17, 29$, we have

$$F_i = \tilde{F}_i, \tag{2.8}$$

where

$$\tilde{F}_i = \begin{cases} p_i - z_i + w_i + 2\,c_{i+1}, & \text{if } i \text{ even or } i = 3, \\ p_i + z_{(i-1)/2} - z_i + w_i + 2\,c_{i+1}, & \text{if } i \text{ odd and } i > 3. \end{cases} \quad (2.9)$$

For the excluded values of i we have

$$F_i = \tilde{F}_i + h_i, \quad (2.10)$$

where

$$\left.\begin{array}{l} h_2 = t_1 + t_2 + t_3 + 2\,t_4 + t_5 + t_6; \;\; h_3 = -t_1 + t_5; \\ h_4 = t_3 + t_6; \;\; h_5 = -t_1 - t_2 - t_3; \;\; h_8 = -t_2; \;\; h_9 = -t_3; \\ h_{11} = t_1; \;\; h_{14} = -t_3; \;\; h_{17} = t_2; \;\; h_{29} = t_3. \end{array}\right\} \quad (2.11)$$

Comparing (2.2) and (2.5) we get the following system of linear algebraic equations in unknowns (2.4):

$$F_i = \sigma_i, \qquad i = 0, 1, 2, \ldots, m. \quad (2.12)$$

Equation $F_i = \sigma_i$ will be denoted by E_i for any non-negative integer i.

The following theorem stems from [3].

Theorem 2.2. *Let L be a symmetric system of numbers of the form $2\cos\frac{p}{q}\pi$, where p, q are integers and $q \neq 0$. A necessary condition for L to be a graph spectrum is that L can be represented in the form* (2.2)*. In this case, to every solution of the system of equations* (2.12) *in unknowns* (2.4)*, these quantities being non-negative integers, a graph corresponds, the spectrum of which is L. All graphs having the spectrum equal to L can be obtained in this way.*

Theorem 2.2 was given in [3] without a detailed proof. Its application requires consideration of some details not mentioned explicitly in the theorem. We shall see that the theorem is valid if equations (2.6)–(2.11) are appropriately specified (see considerations in [4] and in the next sections).

An efficient general theory of systems of linear Diophantine equations does not exist (see, for example, [10]) and therefore we have to use specific features of the system (2.12) when looking for solutions and their properties. However, there are computer tools to handle particular Diofantine equations (for example, package Wolfram MATHEMATICA).

3. *Uniqueness of the canonical representation and an algorithmic criterion for cospectrality of Smith graphs*

Remark 3.1. The quantity m in Theorem 2.1 is bounded by a function $M(n)$ of the number of vertices n. In particular, we have $M(n) = \max\{2n-3, 29\}$ having in view formulas (2.1). The uniqueness of the representation of Theorem 2.1 will be explained below.

We shall first explain in some detail how to find representation (2.2). These arguments will justify the claim that (2.2) is unique and also the uniqueness of the representation of Theorem 2.1.

Suppose we have a symmetric system (family) L of numbers of the form $2\,\cos\frac{p}{q}\pi$ with non-negative multiplicities.

Among $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$ no two systems have the same greatest element. Spectral radius of C_4 is equal to 2 while P_i has spectral radius equal to

$$\lambda_{1,i} = 2\,\cos\frac{1}{i+1}\pi.$$

We first find the multiplicity σ_0 of 2 in L and consider the family $L' = L - \sigma_0\widehat{C}_4$.

The greatest element of L' should be of the form

$$\lambda_{1,m} = 2\,\cos\frac{1}{m+1}\pi$$

and it determines the quantity m in (2.2). If the greatest element is not of this form, the system L is not the spectrum of a graph. Otherwise we consider the new system $L'' = L' - \sigma_m\widehat{P}_m$ where σ_m is the multiplicity of $\lambda_{1,m}$.

Considering always the greatest element we continue identifying paths of canonical representation. We shall either complete successfully this process giving rise to (2.2) or the procedure will fail at some moment.

Note that reduced systems $L'', \ldots$ could contain elements with negative multiplicities. In particular, at some steps the greatest element could have a negative multiplicity and that would mean that the corresponding coefficient σ_j is negative.

Next we shall establish a criterion for cospectality of two Smith graphs.

If $\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n$ are some systems (families) of numbers wth non-negative multiplicities and $\sigma_1, \sigma_2, \ldots, \sigma_n$ integers such that the expression

$$\sigma_1\widehat{S}_1 + \sigma_2\widehat{S}_2 + \cdots + \sigma_n\widehat{S}_n$$

can be calculated in at least one way by successive performing the quoted operations without introducing negative multiplicities, then it defines a system $\widehat{S}$ with non-negative multiplicities and we shall say that $\widehat{S}$ is a linear combination of $\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_n$.

Systems with non-negative multiplicities are useful in describing spectra of Smith graphs.

Formulas (2.1) can be rewritten in the following form:

$$\begin{aligned}
\widehat{W_n} &= \widehat{C}_4 + \widehat{P}_n,\\
\widehat{Z}_n &= -\widehat{P}_n + \widehat{P}_{2n+1} + \widehat{P}_1,\\
\widehat{C}_{2n} &= -2\widehat{P}_1 + \widehat{C}_4 + 2\widehat{P}_{n-1},\\
\widehat{T}_1 &= -\widehat{P}_5 - \widehat{P}_3 + \widehat{P}_{11} + \widehat{P}_2 + \widehat{P}_1,\\
\widehat{T}_2 &= -\widehat{P}_8 - \widehat{P}_5 + \widehat{P}_{17} + \widehat{P}_2 + \widehat{P}_1,\\
\widehat{T}_3 &= -\widehat{P}_{14} - \widehat{P}_9 - \widehat{P}_5 + \widehat{P}_{29} + \widehat{P}_4 + \widehat{P}_2 + \widehat{P}_1,\\
\widehat{T}_4 &= -\widehat{P}_1 + \widehat{C}_4 + 2\widehat{P}_2,\\
\widehat{T}_5 &= -\widehat{P}_1 + \widehat{C}_4 + \widehat{P}_3 + \widehat{P}_2,\\
\widehat{T}_6 &= -\widehat{P}_1 + \widehat{C}_4 + \widehat{P}_4 + \widehat{P}_2.
\end{aligned} \tag{3.1}$$

Given the spectrum of a Smith graph as the sum of spectra of its components, using relations (3.1) we can eliminate left hand side quantities and obtain the spectrum in its canonical form. Since in all formulas (3.1) the sign of the term $\widehat{P}_i$ with the greatest index i is positive, this proves the first assertion of Theorem 2.1.

Let $H \in \mathcal{S}$. Let

$$\widehat{H} = \sigma_0\widehat{C}_4 + \sum_{i=1}^{m} \sigma_i\widehat{P}_i,$$

be the canonical representation of the spectrum $\widehat{H}$ of H. If all quantities σ_i are non-negative, the graph H is called a *Smith graph of type* A, otherwise it is *of type* B. Let I (J) be the set of indices i for which σ_i in a graph of type B is negative (positive).

Let $P_H = \sum_{i\in I} |\sigma_i| P_i$. Components of the graph P_H are paths whose spectra appear with a negative sign in the canonical representation of the spectrum of H. The graph P_H is called the *basis* of H. The basis of a graph of type A is empty. If we add components from its basis to a graph of type B, it becomes a graph of type A.

The graph $K_H = \sigma_0 C_4 + \sum_{i\in J} \sigma_i P_i$ is called the *kernel* of H.

Together with formulas (2.1) we shall consider the corresponding component transformations:

$$
\begin{array}{lcr}
(\gamma_1) & W_n \rightleftarrows C_4 + P_n, & (\delta_1) \\
(\gamma_2) & Z_n + P_n \rightleftarrows P_{2n+1} + P_1, & (\delta_2) \\
(\gamma_3) & C_{2n} + 2P_1 \rightleftarrows C_4 + 2P_{n-1}, & (\delta_3) \\
(\gamma_4) & T_1 + P_5 + P_3 \rightleftarrows P_{11} + P_2 + P_1, & (\delta_4) \\
(\gamma_5) & T_2 + P_8 + P_5 \rightleftarrows P_{17} + P_2 + P_1, & (\delta_5) \\
(\gamma_6) & T_3 + P_{14} + P_9 + P_5 \rightleftarrows P_{29} + P_4 + P_2 + P_1, & (\delta_6) \\
(\gamma_7) & T_4 + P_1 \rightleftarrows C_4 + 2P_2, & (\delta_7) \\
(\gamma_8) & T_5 + P_1 \rightleftarrows C_4 + P_3 + P_2, & (\delta_8) \\
(\gamma_9) & T_6 + P_1 \rightleftarrows C_4 + P_4 + P_2. & (\delta_9)
\end{array}
\tag{3.2}
$$

They are of the form $A \to B$ or $B \to A$ meaning that in a graph the group of components A is replaced with the group of components B or vice versa. Transformations (3.2) are called *G-transformations*. Those of the form $A \to B$ are denoted by (γ_1), (γ_2), …, (γ_9) and are called *C-transformations*. For each C-transformation $A \to B$ we define the corresponding opposite transformation $B \to A$, also denoted by $A \leftarrow B$. Transformations $A \leftarrow B$ are called *D-transformations* and are denoted by (δ_1), (δ_2), …, (δ_9).

Theorem 3.1. *Let H_1 and H_2 be Smith graphs with corresponding bases P_{H_1} and P_{H_2}. If graphs H_1 and H_2 are cospectral, then the graph $H_1 + P_{H_1}$ can be transformed into $H_2 + P_{H_2}$ by a finite number of G-transformations.*

PROOF. If H_1 and H_2 are cospectral, according to Theorem 2.1 their spectrum has the same canonical representation, $P_{H_1} = P_{H_2}$ and $K_{H_1} = K_{H_2}$. By at most 9 of formulas (3.1) the spectrum of H_1 can be reduced to its canonical form. Let $c_1^1, c_2^1, \dots c_u^1$, $u \leq 9$, be the corresponding C-transformations by which $H_1 + P_{H_1}$ is transformed to the kernel of H_1. Let $c_1^2, c_2^2, \dots c_v^2$, $v \leq 9$ be the corresponding C-transformations related to reducing H_2 to the (same) kernel. Let $d_1^2, d_2^2, \dots d_v^2$ be the corresponding D-transformations. Now we can conclude that the sequence of G-transformations $c_1^1, c_2^1, \dots c_u^1$ $d_v^2, \dots, d_2^2, d_1^2$ transforms the graph $H_1 + P_{H_1}$ into graph $H_2 + P_{H_2}$. □

We can use Theorem 3.1 to find the cospectral equivalence class of a Smith graph H. One should start from the graph $H + P_H$ and apply G-transformations whenever possible. By considering all possibilities of application of these transformations we can find all cospectral mates of H. The set of applicable G-transformations is finite.

The described algorithm is an alternative to solving the system of equations (2.12) when looking for the cospectral equivalence class of a Smith graph.

Example 3.1. It was proved in [4] using the extended system of equations that the cospectral equivalence class of the graph $T_5 + T_6$ is equal to the set $\{H_1, H_2, H_3, H_4\}$ where

$$H_1 = T_5 + T_6, H_2 = C_6 + C_4 + P_4 + P_3, H_3 = C_6 + W_3 + P_4, H_4 = C_6 + W_4 + P_3.$$

The algorithmic approach of Theorem 3.1 for this case is illustrated in Fig. 2.

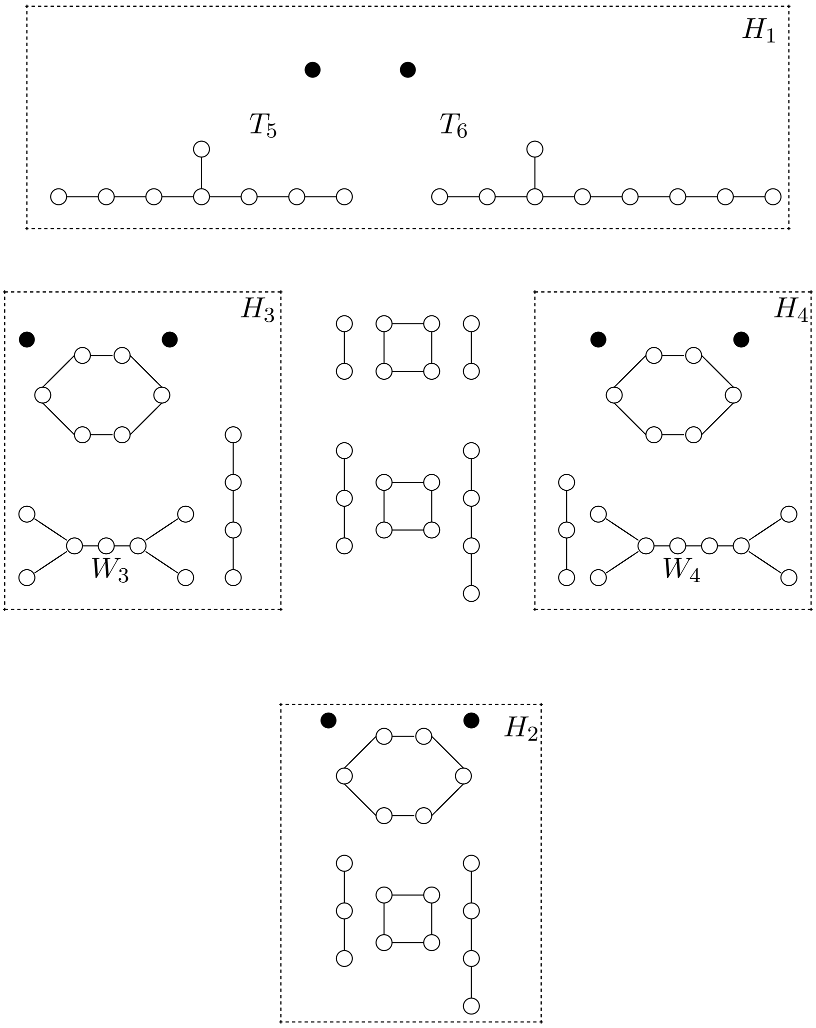

Figure 2. Finding graphs cospectral to $T_5 + T_6$

Using formulas (2.1) we find that

$$\widehat{H}_1 = \widehat{T}_5 + \widehat{T}_6 = 2\widehat{C}_4 - 2\widehat{P}_1 + 2\widehat{P}_2 + \widehat{P}_3 + \widehat{P}_4.$$

Hence we have for H_1 the basis $P_{H_1} = 2P_1$ and the kernel

$$K_{H_1} = 2C_4 + 2P_2 + P_3 + P_4.$$

In Fig. 2 the (common) kernel is placed in the middle. The common basis $2P_1$ (black vertices) is added to each of graphs H_1, H_2, H_3, H_4. Next, we see that $H_1 + 2P_1$ is transformed into the kernel by transformations γ_8 and γ_9. Using transformation δ_3 we replace $C_4 + 2P_2$ into $C_6 + 2P_1$ when passing to all three remaining graphs. Finally, using δ_1 we get $W_3 + P_4$ in H_3 and $W_4 + P_3$ in H_4.

Remark 3.2. In application of Theorem 3.1 the order of performing G-transformation might be sometimes important. This happens in Smith graphs of type A if in forming their canonical forms some terms with negative signs are canceled. For example, we have

$$\widehat{W_1} + \widehat{T}_4 = \widehat{C}_4 + \widehat{P}_1 - \widehat{P}_1 + \widehat{C}_4 + 2\widehat{P}_2 = 2\widehat{C}_4 + 2\widehat{P}_2.$$

In this case, considering $W_1 + P_4$ we should first apply γ_1 to obtain $C_4 + P_1 + T_4$. Now it is possible to apply γ_7 and we get $2C_4 + 2P_2$.

4. *Some general properties of the system of equations*

Remark 4.1. Equality (2.2) can be formulated as

$$L = \sigma_0\widehat{C}_4 + \sum_{i=1}^{+\infty}\sigma_i\widehat{P}_i,$$

with $\sigma_i = 0$ for $i > m$. Together with equations (2.12) we can consider equations $F_i = 0$ for $i > m$ and they also should be fulfilled. Here F_i is defined by (2.9) and (2.10) for any $i > m$. We shall see later that the number of useful equations is still limited. The system (2.12) will be called *basic* and together with additional equations it is called *extended*.

Some examples of application of Theorem 2.2 have been described in [4]. We reproduce here just a simple one.

Example 4.1. Let us find all graphs with the spectrum $L = 2, 0, 0, 0, -2$. We have $L = \widehat{C}_4 + \widehat{P}_1$. The system (2.12) reduces to the equations $w_1 + c_2 = 1$, $p_1 + w_1 + z_2 = 1$ with solutions $w_1 = 0$, $c_2 = 1$, $p_1 = 1$, $z_2 = 0$ and $w_1 = 1$, $c_2 = 0$, $p_1 = 0$, $z_2 = 0$. Hence, graphs $C_4 + P_1$ and W_1 both have the spectrum L. □

Given a bipartite graph G, we can represent it in the canonical form, defined by Theorem 2.1, and find the corresponding canonical coefficients $\sigma_0, \sigma_1, \ldots, \sigma_m$. The corresponding system of equations (2.12) will be called the system *associated* to the graph G. We shall assume in this section that the system we are considering is associated to a graph.

The following proposition has been proved in [4].

Proposition 4.1. *If $\sigma_0, \sigma_1, \ldots, \sigma_m$ are coefficients of the canonical representation of the spectrum of a bipartite graph G from $\mathcal{S}$, then the number n of vertices of G is given by*

$$n = 4\sigma_0 + \sum_{i=1}^{m} i\, \sigma_i.$$

Example 4.2. Based on equations (2.1) we have the following canonical forms for the spectra of Z_n and T_3 respectively:

$$\widehat{Z_n} = \widehat{P_1} - \widehat{P_n} + \widehat{P}_{2n+1},$$

$$\widehat{T_3} = \widehat{P_1} + \widehat{P_2} + \widehat{P_4} - \widehat{P_5} - \widehat{P_9} - \widehat{P}_{14} + \widehat{P}_{29}.$$

By Proposition 4.1 we have for the number of vertices $1 - n + 2n + 1 = n + 2$ for Z_n and $1 + 2 + 4 - 5 - 9 - 14 + 29 = 8$ for T_3.

From Proposition 4.1 we conclude that the number n of vertices of unknown graphs is uniquely determined by the system of equations.

The number n determines the set of variables in the system (2.12). One should include variables indicating the number of components whose number of vertices is at most n.

Example 4.3. For considering graphs on 6 vertices the following variables are relevant: $p_1, p_2, p_3, p_4, p_5, p_6; z_2, z_3, z_4; w_1, w_2; c_2, c_3$ and t_1.

If we take 6 equations, the matrix of the system (2.12) reads:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.$$

Let us establish the number of variables.

Remark 4.2. Given n the number of vertices the following variables are relevant $p_1, p_2, \ldots, p_n$; $z_2, z_3, \ldots, z_{n-2}$; $w_1, w_2, \ldots, w_{n-4}$; $c_2, c_3, \ldots, c_{[n/2]}$ and $t_1, t_2, t_3, t_4, t_5, t_6$.

Let us define τ_n for $n \geq 5$ by the following table.

n	5	6	7	8	≥ 9
τ_n	0	1	3	5	6

We have for $n \geq 5$ counting in turn $n + (n-3) + (n-4) + [n/2] - 1 + \tau_n = 3n + [n/2] - 8 + \tau_n$ variables.

We shall always assume that $n \geq 5$ since otherwise the system is not interesting (in particular, the smallest number of vertices in non-isomorphic cospectral graphs is 5).

We shall see later that the list of relevant variables can be reduced. □

Let $v_1, v_2, \ldots, v_s$ are variables of our system. Let for any i the number of vertices in the corresponding component of the considered graph be denoted by $N(v_i)$. In particular, we have $N(p_j) = j$, $N(z_j) = j + 2$, $N(w_j) = j + 4$ and $N(c_j) = 2j$ for any suitable j. Then

$$N(v_1) + N(v_2) + \cdots + N(v_s) = n.$$

This equation should be added to the system since this makes finding solutions easier. It will be denoted by E_v.

Remark 4.3. The system (2.12) always has a solution $c_2 = \sigma_0$, $p_1 = \sigma_1$, …, $p_m = \sigma_m$ with other variables being equal to 0, giving rise to a hypothetical graph $\sigma_0 C_4 + \sigma_1 P_1 + \sigma_2 P_2 + \cdots + \sigma_m P_m$. However, this formal linear combination does not correspond to a graph if among coefficients σ_i are some which are negative. In this case we know that still a solution exists since we assume that the system is associated to a graph G. This solution is expressed through parameters of G. Such a solution is called *standard solution* of system (2.12). Obviously, a graph G is a DS-graph if and only if the system (2.12), associated to G, has a unique solution (i.e., only standard solution). In the contrary, in order to determine the cospectral equivalence class of some non DS-graph, we are interested in non-standard solutions of the associated system. Obviously, a graph G is DS if and only if the system (2.12), associated to G, has only the standard solution.

5. *Some details concerning the extended system of equations*

The purpose of this section is to represent in a clear way equations from the basic and from the extended system of equations.

Variables $t_1, t_2, t_3, t_4, t_5, t_6$ will be called *exceptional*. Let

$$T = t_1 + t_2 + t_3 + t_4 + t_5 + t_6.$$

We shall first consider our system for which $T = 0$. The system becomes simpler and we can analyze it easier. Afterwards we shall consider the case $T > 0$.

Equations E_i for $i > n$ have a simple form. From (2.9) we have

$$F_{2p} = 0, \quad F_{2p+1} = z_p.$$

Since maximal value of p is $n-2$, we see that the equation

$$E_{2n-3} : \; F_{2n-3} = z_{n-2}$$

is the one with the largest index i in E_i that should be considered. Equations E_i for $i > 2n-3$ are of the form $0 = 0$.

Now we see that our system is reduced to equations

$$E_v, E_0, E_1, E_2, \ldots, E_{2n-3}.$$

These equations have been considered in [4] when determining cospectral equivalence classes for graphs $W_1 + T_4$, $W_1 + T_5$ and $T_5 + T_6$.

However, equations E_i with i even and $n < i \leq 2n-3$ are useless since they are of the form $0 = 0$.

Equations E_{2p+1} for $n < 2p+1 \leq 2n-3$ contain only the variable z_p. These variables can be immediately determined and eliminated from the rest of the system. Note that only one of these variables can be equal to 1, other being equal to 0. Therefore the system is reduced to equations

$$E_v, E_0, E_1, E_2, \ldots, E_n.$$

After all these reductions the system has the following form (we quote left hand sides F_i of the corresponding equations):

$$F_v = N(v_1) + N(v_2) + \cdots + N(v_s) \; (= n),$$
$$F_0 = (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots),$$
$$F_1 = p_1 + w_1 + (z_2 + z_3 + \cdots) - 2\,(c_3 + c_4 + \cdots),$$
$$F_2 = p_2 - z_2 + w_2 + 2\,c_3,$$
$$F_3 = p_3 - z_3 + w_3 + 2\,c_4,$$
$$F_4 = p_4 - z_4 + w_4 + 2\,c_5,$$
$$F_5 = p_5 + z_2 - z_5 + w_5 + 2\,c_6,$$

if i is even: $F_i = p_i - z_i + w_i + 2\,c_{i+1}$,	if i is odd: $F_i = p_i + z_{\frac{i-1}{2}} - z_i + w_i + 2\,c_{i+1}$
up to $i = [n/2] - 1$	
if i is even: $F_i = p_i + w_i$,	if i is odd: $F_i = p_i + z_{\frac{i-1}{2}} + w_i$,
for $[n/2] - 1 < i \leq n - 4$	
if n is even: $F_{n-3} = p_{n-3} + z_{(n-4)/2}$, $F_{n-2} = p_{n-2}$, $F_{n-1} = p_{n-1} + z_{(n-2)/2}$, $F_n = p_n$.	if n is odd: $F_{n-3} = p_{n-3}$, $F_{n-2} = p_{n-2} + z_{(n-3)/2}$, $F_{n-1} = p_{n-1}$, $F_n = p_n + z_{(n-1)/2}$.

We see that most of equations contain a small number of variables. Starting from E_n, one should be able to determine immediately a lot of variables (see next section).

Exceptional variables $t_1, t_2, t_3, t_4, t_5, t_6$ appear in equation E_v and in equations E_i for $i = 0, 1, 2, 3, 4, 5, 8.9, 11, 14, 17, 29$. Last 12 equations can be presented in the form

$$\begin{array}{rrrrrrcl}
 & & & t_4 & +t_5 & +t_6 & = & a_0, \\
t_1 & +t_2 & +t_3 & -t_4 & -t_5 & -t_6 & = & a_1, \\
t_1 & +t_2 & +t_3 & +2t_4 & +t_5 & +t_6 & = & a_2, \\
 & & & -t_1 & +t_5 & & = & a_3, \\
 & & t_3 & & & +t_6 & = & a_4, \\
-t_1 & -t_2 & -t_3 & & & & = & a_5, \\
 & -t_2 & & & & & = & a_8, \\
 & & -t_3 & & & & = & a_9, \\
t_1 & & & & & & = & a_{11}, \\
 & & -t_3 & & & & = & a_{14}, \\
 & t_2 & & & & & = & a_{17}, \\
 & & t_3 & & & & = & a_{29}.
\end{array}$$

In equation E_i all terms different from t_i's are collected on the right hand side with mark a_i.

For the convenience of the reader we shall repeat equations $E_0 - E_5$ with added exceptional variables.

$$F_0 = (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots) + t_4 + t_5 + t_6,$$
$$F_1 = p_1 + w_1 + (z_2 + z_3 + \cdots) - 2\,(c_3 + c_4 + \cdots) + t_1 + t_2 + t_3 - t_4 - t_5 - t_6,$$
$$F_2 = p_2 - z_2 + w_2 + 2\,c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6,$$
$$F_3 = p_3 - z_3 + w_3 + 2\,c_4 - t_1 + t_5,$$
$$F_4 = p_4 - z_4 + w_4 + 2\,c_5 + t_3 + t_6,$$
$$F_5 = p_5 + z_2 - z_5 + w_5 + 2\,c_6 - t_1 - t_2 - t_3.$$

6. *Reduction of the system*

Consider a system of equations

$$E_v, E_0, E_1, E_2, \ldots, E_n, \ldots, E_q$$

associated to a bipartite Smith graph on n vertices. The system contains relevant variables as listed in Remark 4.2. This is the extended system of equations for spectra of Smith graphs. Equations E_i for $i > q$ are of the form $0 = 0$ while equation E_q contains at least one variable. Of course, $m \le q = \max\{2n-3, 29\}$.

We shall consider these equations in the direction “bottom - up”, i.e., from E_q up to E_m.

Lemma 6.1. *Let v be any of variables p_j, z_j, w_j, c_j for some j or t_1, t_2, t_3. The variable v appears with sign + in the “lowest” equation E_i in which it appears.*

PROOF. The statement follows from formulas (2.1) and from the way in which equations E_i are constructed. □

Theorem 6.1. *When solving the extended system, one can restrict to the following equations $E_v, E_0, E_1, \ldots, E_m$.*

PROOF. By definition of the parameter m, the equation E_i is of the form $F_i = 0$ for $i > m$ and if F_i contains the sum of non-negative variables, all they have to be equal 0. By Lemma 6.1, this happens in equation E_q. We consider the system of equations in the direction bottom – up, from E_q up to E_m. In the moment when we consider $F_i = 0$ containing a variable v with $-$ sign, then v is already determined as equal to 0 (when we were considering one of the equations $F_j = 0$, $j > i$). In this way, we establish that all variables from equations $E_q, E_{q-1}, \ldots, E_{m+1}$ are equal to 0. This proves the theorem. □

When reducing the system of equations, the original set of variables from Remark 4.2 is also reduced. For a special case we can prove the following theorem.

Theorem 6.2. *When solving the extended system with* $T = 0$ *and* $5 \leq m \leq [n/2] - 1$*, one can restrict to the equations* $E_v, E_0, E_1, \ldots, E_m$ *with the following variables:* $p_1, p_2, \ldots, p_m$*;* $w_1, w_2, \ldots, w_m$*;* $c_2, c_3, \ldots, c_{m+1}$ *and* $z_2, z_3, \ldots, z_t$*, where* $t = m/2 - 1$ *for* m *even and* $t = (m-1)/2$ *for* m *odd.*

PROOF. As in the proof of Theorem 6.1, we establish that all variables from equations $E_q, E_{q-1}, \ldots, E_{m+1}$ are equal to 0. When considering E_{m+1} we establish that the following variables are equal to 0: $p_{m+1}, w_{m+1}, c_{m+2}$ and $z_{m/2}$ for m even and $z_{(m+1)/2}$ for m odd. This proves the theorem. □

By proving Theorem 6.2 the meaning of Theorem 2.2 becomes more precise since it was not clear what variables really take part in the system (2.12).

In each particular case one can establish exactly which variables remain.

Theorem 6.2 remains valid for $m < 5$ in which case no of variables $z_2, z_3, \ldots$ appears after reduction of the system.

Without condition $T = 0$, if $m \leq 10$, we can conclude that $t_1, t_2, t_3 = 0$. If $m \leq 3$ then from E_4 we get also $t_6 = 0$ and if $m \leq 2$ we conclude from E_3 that $t_5 = 0$.

Example 6.1. The cospectral equivalence class of graph $W_1 + T_4$ consists of the following seven graphs: $W_1 + T_4$, $P_1 + C_6 + W_1$, $P_1 + C_4 + T_4$, $P_2 + C_4 + W_2$, $2P_2 + 2C_4$, $2W_2$ and $C_6 + C_4 + 2P_1$. This was proved in [4] using extended system of equations. Indeed, graph $W_1 + T_4$ has 12 vertices, and we have: $\widehat{W}_1 + \widehat{T}_4 = 2\widehat{C}_4 + 2\widehat{P}_2$. This means that $m = 2$. Using Theorem 6.1 and above remarks, it is sufficient to consider the following equations:

$$\begin{array}{lclcccccccccccc} F_0 & = & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & 2, \\ F_1 & = & p_1 & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & 0, \\ F_2 & = & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & 2. \end{array}$$

Equation E_2 has five solutions and these readily provide seven solutions of the system, as described in [4].

7. *Special cases*

***Height m* = 0.** We have $m = 0$ and by bottom-up principle all variables are equal to 0 except for c_2 in E_0. In fact, $F_0 = c_2 = \sigma_0$ and the solution is unique: $\sigma_0 C_4$. More general result is well known: regular graphs of degree 2 are DS (cf. [2], p. 167).

***Height m* = 1.** The extended system is reduced to $w_1 + c_2 = \sigma_0, p_1 + w_1 = \sigma_1$ with solutions $w_1 = k, c_2 = \sigma_0 - k, p_1 = \sigma_1 - k$ for $0 \leq k \leq \min\{\sigma_0, \sigma_1\}$. We have here a slight generalization of Example 4.1.

Spectral characterization of connected Smith graphs. It is known from the literature that connected Smith graphs are DS except for W_n and T_4 (cf. relations (2.1)). W_n is cospectral with $C_4 + P_n$ and T_4 is cospectral with $C_6 + P_1$. We can confirm these results using our technique and will give here only a few examples.

For P_n the extended system is reduced to the equation $F_n = p_n = 1$ proving that P_n is DS.

For T_1 we have $n = 6$ and $m = 11$. We immediately get $F_{11} = t_1 = 1$ as required. For T_2 and T_3 relevant equations are $F_{17} = t_2 = 1$ and $F_{29} = t_3 = 1$ respectively.

Of course, these tricks will not work for T_4. We have $n = 7, m = 2$ and the following equations

$$\begin{array}{lcccccccccccccc}
F_0 & = & & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & 1, \\
F_1 & = & p_1 & & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & -1, \\
F_2 & = & & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & 2.
\end{array}$$

From E_2 we get for non-zero variables either $t_4 = 1$ or $c_3 = 1$ since $p_2 = w_2 = 1$ yields a graph on 8 vertices. Hence we readily get what is expected. Note that expressions for F_0, F_1, F_2 are the same as in Example 6.1.

We have proved in [4] that the graph Z_n is DS using our technique but the result was known in the literature, obtained by other techniques.

***Height m* = 2.** We already had two examples with $m = 2$. Now we formulate the general case where we add equation E_v as well.

$$\begin{array}{lcccccccccccccccccc}
F_v & = & p_1 & + & 2p_2 & + & 5w_1 & + & 6w_2 & + & 4c_2 & + & 6c_3 & + & 7t_4 & = & n, \\
F_0 & = & & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & \sigma_0, \\
F_1 & = & p_1 & & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & \sigma_1, \\
F_2 & = & & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & \sigma_2.
\end{array}$$

By Proposition 4.1 we have $n = 4\sigma_0 + \sigma_1 + 2\sigma_2$, where n is the number of vertices.

Appendix: Spectra of Smith graphs

We shall list spectra of Smith graphs as they are given in [3].

$P_n: \; 2\cos\frac{j\pi}{n+1}, \quad j = 1, 2, \ldots, n,$

$Z_n: \; 2\cos\frac{(2j+1)\pi}{2(n+1)}, \quad j = 0, 1, \ldots, n,$ and 0,

$W_n: \; 2\cos\frac{j\pi}{n+1}, \quad j = 1, 2, \ldots, n,$ and $2, 0, 0, -2,$

$C_n: \; 2\cos\frac{2j\pi}{n}, \quad j = 1, 2, \ldots, n,$

$T_1: \; 2\cos\frac{j\pi}{12}, \quad j = 1, 4, 5, 7, 8, 11,$

$T_2: \; 2\cos\frac{j\pi}{18}, \quad j = 1, 5, 7, 9, 11, 13, 17,$

$T_3: \; 2\cos\frac{j\pi}{30}, \quad j = 1, 7, 11, 13, 17, 19, 23, 29,$

$T_4: \; 2\cos\frac{2j\pi}{6}, \quad j = 1, 2, 3, 4, 5, 6,$ and 0,

$T_5: \; 2\cos\frac{j\pi}{4}, \quad j = 1, 2, 3,$ and $2, 1, 0, -1, -2,$

$T_6: \; 2\cos\frac{j\pi}{5}, \quad j = 1, 2, 3, 4,$ and $2, 1, 0, -1, -2.$

Spectra of P_n, Z_n, W_n and C_n had been known before publication of [3]. Relevant references can be found in [3] and [2], p. 77. Spectra of $T_1 - T_6$ have been given in [3] with the remark that they can be obtained "by direct calculation, although this is not simple in all cases".

One way to verify spectra of $T_1 - T_6$ is to use characteristic polynomials of graphs. Let $\widetilde{G}$ be the characteristic polynomials of the graph G. We have (cf. [2], p. 77)

$$\widetilde{P}_n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{n-2k}.$$

Characteristic polynomials of $T_1 - T_6$ can be reduced to characteristic polynomials of paths using Theorem 2.11 from [2]. Alternatively, they can be found in tables of trees up to 10 vertices from [2]. In particular, we have

$$\widetilde{T}_1 = x^6 - 5x^4 + 5x^2 - 1.$$

Relations (2.1) can be rewritten in terms of characteristic polynomials. For example, the fourth relation (2.1) yields $\widetilde{T_1}\widetilde{P_5}\widetilde{P_3} = \widetilde{P_{11}}\widetilde{P_2}\widetilde{P_1}$, which can be directly verified. In this way, the verification of spectra of $T_1 - T_6$ is performed by multiplication of polynomials.

Alternatively, characteristic equations $\widetilde{T_i} = 0$, $i = 1, 2, \dots, 6$, can be reduced to trigonometric equations by the substitution $x = 2\cos t$, as actually done when preparing [3].

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