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ON SOME INTEGRALS INVOLVING $\Delta_2(x)$ AND $\Delta_3(x)$

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In memory of Professor Aleksandar Ivić

(Presented at the 2nd Meeting, held on March 26, 2021)

A b s t r a c t. Let $k \geq 2$ be a fixed natural number and $d_k(n)$ denote the number of ways n can be written as a product of k positive integers. Let $\Delta_k(x)$ denote the error term in the asymptotic formula of the summatory function of $d_k(n)$. In this paper we study some integrals involving $\Delta_2(x)$ and $\Delta_3(x)$.

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1. *Introduction and statements of results*

Let $k \geq 2$ be a fixed natural number and $d_k(n)$ denote the number of ways n can be written as a product of k positive integers. The general divisor problem is to study the asymptotic behaviour of the sum

$$D_k(x) := \sum_{n \leq x} d_k(n)$$

as x tends to infinity. It is well-known that the asymptotic formula of $D_k(x)$ is of the form

$$D_k(x) = xP_k(\log x) + \Delta_k(x), \tag{1.1}$$

where $P_k(u)$ is a polynomial in u of degree $k-1$, and $\Delta_k(x)$ is the error term. It is a classical and important problem in the analytic number theory to study properties of $\Delta_k(x)$.

When $k=2$, this is usually called the Dirichlet divisor problem since Dirichlet first proved that $\Delta_2(x) \ll \sqrt{x}$. Dirichlet's result was improved by many authors. The best upper bound up to date is $\Delta_2(x) \ll x^{517/1648+\varepsilon}$ proved in Bourgain and Watt [1]. It is conjectured that the estimate $\Delta_2(x) \ll x^{1/4+\varepsilon}$ holds, which is supported by the mean square result(see Cramér [3])

$$\int_1^T (\Delta_2(x))^2 \, \mathrm{d}x = C_2^{(2)} T^{3/2} + O(T^{5/4+\varepsilon}) \tag{1.2}$$

and the estimate (see Ivić [5], Thm 13.9)

$$\int_1^T |\Delta_2(x)|^{A_0} \, \mathrm{d}x \ll T^{1+A_0/4+\varepsilon} \;\; (A_0 = 8.75), \tag{1.3}$$

where

$$C_2^{(2)} := \frac{1}{6\pi^2} \sum_{n=1}^{\infty} \frac{d_2^2(n)}{n^{3/2}}.$$

The estimate $T^{5/4+\varepsilon}$ was improved in Tong[14], Preissmann[12], Lau and Tsang [10], respectively. For any integer $3 \le l \le 9$, we have the asymptotic formula

$$\int_1^T (\Delta_2(x))^l \, \mathrm{d}x = C_2^{(l)} T^{1+l/4} + O(T^{1+l/4-\delta_l+\varepsilon}), \tag{1.4}$$

where $C_2^{(l)}$ and $\delta_l > 0$ are explicit constants. Tsang [15] first proved the asymptotic formula (1.4) for $l=3$ and $l=4$ with $\delta_3 = 1/14$ and $\delta_4 = 1/23$. Ivić and Sargos [6] proved the asymptotic formula (1.4) for $l=3$ and $l=4$ with $\delta_3 = 7/20$ and $\delta_4 = 1/12$. Zhai [17] proved (1.4) for any integer $3 \le l \le 9$ by a unified approach. For more results about (1.4), see [9, 11, 16, 19, 20].

When $k=3$, this is usually called the Piltz divisor problem. The best estimate of $\Delta_3(x)$ up to date is $\Delta_3(x) \ll x^{43/96+\varepsilon}$ proved in Kolesnik [8]. Tong [14] proved the mean square result

$$\int_1^T (\Delta_3(x))^2 \, \mathrm{d}x = C_3^{(2)} T^{5/3} + O(T^{14/9+\varepsilon}), \tag{1.5}$$

where

$$C_3^{(2)} := \frac{1}{10\pi^2} \sum_{n=1}^{\infty} \frac{d_3^2(n)}{n^{4/3}}.$$

Heath-Brown[4] proved that the estimate

$$\int_1^T |\Delta_3(x)|^3 \,\mathrm{d}x \ll T^{2+\varepsilon} \tag{1.6}$$

holds.

Ivić and Zhai [7] first studied the hybrid moments of $\Delta_2(x)$ and $\Delta_3(x)$. They proved that

$$\int_1^T \Delta_2(x)\Delta_3(x) \,\mathrm{d}x \ll T^{13/9} \log^{10/3} T. \tag{1.7}$$

In this paper we shall study the upper bound of the integral

$$\int_1^T \Delta_2^k(x)\Delta_3(x) \,\mathrm{d}x$$

with $k = 2$ and $k = 3$. We have the following theorems.

Theorem 1.1. *We have the estimate*

$$\int_1^T \Delta_2^2(x)\Delta_3(x) \,\mathrm{d}x \ll T^{16/9+\varepsilon}. \tag{1.8}$$

Theorem 1.2. *We have the estimate*

$$\int_1^T \Delta_2^3(x)\Delta_3(x) \,\mathrm{d}x \ll T^{73/36+\varepsilon}. \tag{1.9}$$

Remark 1.1. From (1.3), (1.5) and Cauchy's inequality we get

$$\int_1^T \Delta_2^2(x)\Delta_3(x) \,\mathrm{d}x \ll T^{11/6+\varepsilon}, \quad \int_1^T \Delta_2^3(x)\Delta_3(x) \,\mathrm{d}x \ll T^{25/12+\varepsilon}.$$

Note that $16/9 = 11/6 - 1/18,\ 73/36 = 25/12 - 1/18$. So Theorem 1.1 and Theorem 1.2 are non-trivial.

Notation. For any $k \geq 2$, $d_k(n)$ denotes the number of ways n can be written as a product of k natural numbers. $n \sim N$ means $N < n \leq 2N$ and $n \asymp N$ means $c_1 N < n \leq c_2 N$ for absolute positive constants $c_2 > c_1 > 0$. ε always denotes a small positive constant which may be different at different places. For any complex number α, we define $e(\alpha) := \exp(2\pi \mathrm{i} \alpha)$.

2. *Some preliminary lemmas*

In order to prove our theorems, we need the following lemmas.

Lemma 2.1. *Let $c > 0$ be a non-integer real number, $M \geq 2$ be a large parameter, $\delta > 0$ be any real number. Let $\mathcal{A}(M, \delta; c)$ denote the number of solutions of the inequality*

$$|m_1^c + m_2^c - m_3^c - m_4^c| \leq \delta M^c, \quad M < m_1, m_2, m_3, m_4 \leq 2M.$$

Then we have

$$\mathcal{A}(M, \delta) \ll M^{\varepsilon}(M^2 + \delta M^4).$$

PROOF. This is Theorem 2 of [13]. □

Lemma 2.2. *Let $T \geq 10$ be a large parameter and y be a real number such that $T^{\varepsilon} \ll y \ll T$. For any $T \leq x \leq 2T$ define*

$$\Delta_{31}(x; y) := \frac{x^{1/3}}{\sqrt{3}\pi} \sum_{n \leq y} \frac{d_3(n)}{n^{2/3}} \cos(6\pi(nx)^{1/3}).$$

Then uniformly for $y \ll T^{1/3}$ we have

$$\int_T^{2T} |\Delta_{31}(x; y)|^4 \,\mathrm{d}x \ll T^{7/3+\varepsilon}. \tag{2.1}$$

PROOF. By a splitting argument and Hölder's inequality we get for some $1 \ll N \ll y$ that

$$\begin{aligned}
\int_T^{2T} |\Delta_{31}(x; y)|^4 \,\mathrm{d}x &= \int_T^{2T} \left| \frac{x^{1/3}}{\sqrt{3}\pi} \sum_j \sum_{\frac{y}{2^{j+1}} < n \leq \frac{y}{2^j}} \frac{d_3(n)}{n^{2/3}} \cos(6\pi(nx)^{1/3}) \right|^4 \mathrm{d}x \\
&\ll T^{4/3} \log^3 T \sum_j \int_T^{2T} \left| \sum_{\frac{y}{2^{j+1}} < n \leq \frac{y}{2^j}} \frac{d_3(n)}{n^{2/3}} \cos(6\pi(nx)^{1/3}) \right|^4 \mathrm{d}x \\
&\ll T^{4/3} \log^4 T \sum_{n_1, n_2, n_3, n_4 \sim N} \frac{\prod_{j=1}^4 d_3(n_j)}{\prod_{j=1}^4 n_j^{2/3}} \int_T^{2T} e(3x^{1/3}\eta) \,\mathrm{d}x,
\end{aligned}$$

where $\eta := n_1^{1/3} + n_2^{1/3} - n_3^{1/3} - n_4^{1/3}$. By the first derivative test we have

$$\int_T^{2T} e(3x^{1/3}\eta) \,\mathrm{d}x \ll \min\left(T, \frac{T^{2/3}}{|\eta|}\right).$$

So from the above two estimates we have

$$\int_T^{2T} |\Delta_{31}(x;y)|^4 \,\mathrm{d}x \ll \frac{T^{2+\varepsilon}}{N^{8/3}} \sum_{\substack{n_j \sim N \\ j=1,2,3,4}} \min\left(T^{1/3}, \frac{1}{|\eta|}\right). \tag{2.2}$$

According to Lemma 2.1, the contribution of $T^{1/3}$ (in this case $|\eta| \le T^{-1/3}$) is

$$\ll \frac{T^{7/3+\varepsilon}}{N^{8/3}}(N^2 + T^{-1/3}N^{-1/3}N^4) \ll T^{7/3+\varepsilon}N^{-2/3} + T^{2+\varepsilon}N \ll T^{7/3+\varepsilon}$$

if $y \ll T^{1/3}$.

Now we consider the contribution of $1/|\eta|$ for which $T^{-1/3} \ll |\eta| \ll y^{1/3}$. We divide the range of η into $O(\log T)$ subcases of the form $\xi < |\eta| \le 2\xi$. By Lemma 2.1 again we get the the contribution of $1/|\eta|$ is

$$\begin{aligned} &\ll \frac{T^{2+\varepsilon}}{N^{8/3}} \max_{T^{-1/3} \ll \xi \ll y^{1/3}} \sum_{\xi < |\eta| \le 2\xi} \frac{1}{|\eta|} \\ &\ll \frac{T^{2+\varepsilon}}{N^{8/3}} \max_{T^{-1/3} \ll \xi \ll y^{1/3}} \frac{1}{\xi}(N^2 + \xi N^{-1/3}N^4) \\ &\ll T^{7/3+\varepsilon}N^{-2/3} + T^{2+\varepsilon}N \ll T^{7/3+\varepsilon}. \end{aligned}$$

Whence Lemma 2.2 follows. □

Lemma 2.3. *Let $T \ge 10$ be a large parameter and y be a real number such that $T^\varepsilon \ll y \ll T$. Define for any $T \le x \le 2T$ that*

$$\Delta_{32}(x;y) := \Delta_3(x) - \Delta_{31}(x;y).$$

Then we have

$$\int_T^{2T} |\Delta_{32}(x;y)|^2 \,\mathrm{d}x \ll T^{5/3+\varepsilon}y^{-1/3} \quad (y \ll T^{1/3}). \tag{2.3}$$

PROOF. We give a very short proof (2.3). We take $d = 3$, $a(n) = d_3(n)$, $N = [T^{5-\varepsilon}]$, $\sigma^* = 7/12$, and $M = [T^{2/3}]$. As in the proof of Theorem 1 in [2], we can write

$$\Delta_{32}(x;y) = R_1^*(x;y) + \sum_{j=2}^{7} R_j(x),$$

where

$$R_1^*(x;y) := \frac{x^{1/3}}{\sqrt{3}\pi} \sum_{y<n\le M} \frac{d_3(n)}{n^{2/3}} \cos(6\pi(nx)^{1/3})$$

and $R_j(x)$ $(j = 2,3,4,5,6,7)$ were defined in Page 2129 of [2]. Similar to the formula (8.11) of [2], we have the estimate (noting that $y \ll T^{1/3}$)

$$\begin{aligned}\int_T^{2T} &(R_1^*(x;y)+R_2(x))^2\,\mathrm{d}x\\ &\ll \sum_{y<n<M} \frac{d_3^2(n)}{n^{4/3}} \int_T^{2T} x^{2/3}\,\mathrm{d}x + T^{5/3+\varepsilon}M^{-1/6} + T^{4/3+\varepsilon}M^{1/3}\\ &\ll T^{5/3+\varepsilon}y^{-1/3} + T^{14/9+\varepsilon} \ll T^{5/3+\varepsilon}y^{-1/3},\end{aligned}$$

which combining (8.17) of of [2] gives (2.3). □

Lemma 2.4. *Let $T \ge 10$ be a large parameter and Y be a real number such that $T^\varepsilon \ll Y \ll T$. For any $T \le x \le 2T$ define*

$$\Delta_{21}(x;Y) := \frac{x^{1/4}}{\sqrt{2}\pi} \sum_{n\le Y} \frac{d_2(n)}{n^{3/4}} \cos\left[4\pi(nx)^{1/2} - \frac{\pi}{4}\right].$$

Then uniformly for $Y \ll T$ we have

$$\int_T^{2T} |\Delta_{21}(x;Y)|^4\,\mathrm{d}x \ll T^{2+\varepsilon}. \tag{2.4}$$

If $Y \ll T^{1/3}$, then we have

$$\int_T^{2T} |\Delta_{21}(x;Y)|^8\,\mathrm{d}x \ll T^{3+\varepsilon}. \tag{2.5}$$

PROOF. For any $1 \ll N \ll x$, we have (see, for example, [5])

$$\Delta_2(x) = \Delta_{21}(x;N) + O\left(x^{1/2+\varepsilon}N^{-1/2}\right). \tag{2.6}$$

We first show (2.4). If $Y \ll T^{1/2}$, then by the same approach as the proof of Lemma 2.2 we get (2.4). If $T \gg T^{1/2}$, from (2.6) we have

$$\Delta_{21}(x;Y) = \Delta_{21}(x;\sqrt{T}) + O(x^{1/4+\varepsilon}).$$

So (2.4) follows from the case $y = \sqrt{T}$.

Using the approaches of Theorems 13.8 and 13.9 directly to $\Delta_{21}(x;Y)$ we can get (2.5) easily. So, we omit the details. □

Lemma 2.5. *Let $T \geq 10$ be a large parameter and Y be a real number such that $T^{\varepsilon} \ll Y \ll T$. For any $T \leq x \leq 2T$ define*

$$\Delta_{22}(x;Y) := \Delta_2(x) - \Delta_{21}(x;Y).$$

Then for $Y \ll T$ we have

$$\int_T^{2T} |\Delta_{22}(x;Y)|^2 \,\mathrm{d}x \ll T^{3/2+\varepsilon} Y^{-1/2}. \tag{2.7}$$

If $Y \ll T^{1/3}$, then we have

$$\int_T^{2T} |\Delta_{22}(x;Y)|^4 \,\mathrm{d}x \ll T^{2+\varepsilon} Y^{-1/3}. \tag{2.8}$$

PROOF. The estimate (2.7) is Lemma 2.2 of Ivić and Zhai [7].

From (2.5) and (1.3) with $A_0 = 8$ we have

$$\begin{aligned}\int_T^{2T} |\Delta_{22}(x;Y)|^8 \,\mathrm{d}x &= \int_T^{2T} |\Delta_2(x) - \Delta_{21}(x;Y)|^8 \,\mathrm{d}x \\ &\ll \int_T^{2T} |\Delta_{21}(x;Y)|^8 \,\mathrm{d}x + \int_T^{2T} |\Delta_2(x)|^8 \,\mathrm{d}x \ll T^{3+\varepsilon}.\end{aligned} \tag{2.9}$$

Now from (2.7), (2.9) and Hölder's inequality we get

$$\begin{aligned}\int_T^{2T} |\Delta_{22}(x;Y)|^4 \,\mathrm{d}x &\ll \left(\int_T^{2T} |\Delta_{22}(x;Y)|^2\right)^{2/3} \left(\int_T^{2T} |\Delta_{22}(x;Y)|^8\right)^{1/3} \\ &\ll T^{2+\varepsilon} Y^{-1/3}.\end{aligned}$$

□

Lemma 2.6. *Suppose $(i_1, i_2) \in \{0,1\}^2$ and $Y \geq 10$ is a real number. For $(n_1, n_2, n_3) \in \mathbb{N}^3$, define*

$$\alpha_3 : = \sqrt{n_1} + (-1)^{i_1}\sqrt{n_2} + (-1)^{i_2}\sqrt{n_3},$$

$$H(Y; i_1, i_2) : = \sum_{\substack{n_j \leq Y, 1 \leq j \leq 3 \\ \alpha_3 \neq 0}} \frac{d_2(n_1) d_2(n_2) d_2(n_3)}{(n_1 n_2 n_3)^{3/4} |\alpha_3|}.$$

Then we have

$$H(Y;i_1,i_2) \ll Y^{1/4+\varepsilon}.$$

PROOF. This is Lemma 2.6 of the author [18]. □

Lemma 2.7. *Let* $N, M, K \geq 10$, $D = \max(N,M,K)$, $0 < \Delta \ll D^{1/2}$. *Let*

$$\mathcal{A}(N,M,K;\Delta) := \sum_{\substack{n\sim N, m\sim M, k\sim K \\ |\sqrt{n}+\sqrt{m}-\sqrt{k}|\leq \Delta}} 1.$$

Then we have

$$D^{-\varepsilon}\mathcal{A}(N,M,K;\Delta) \ll \Delta D^{-1/2}NMK + D^{-1/2}(NMK)^{1/2}.$$

PROOF. This is Lemma 2.4 of the author [18]. □

Lemma 2.8. *Suppose* $f(x)$ *is real-valued such that* $|f'(x)| \gg \Delta > 0$ *in* $[a,b]$, *then we have*

$$\int_a^b e(f(x))\,\mathrm{d}x \ll \frac{1}{\Delta}.$$

PROOF. See, for example, Ivić [5]. □

Lemma 2.9. *We have the estimates*

$$\sum_{n\leq x} d_2(n) \ll x\log x, \quad \sum_{n\leq x} d_3(n) \ll x\log^2 x.$$

PROOF. See, for example, Ivić [5]. □

Lemma 2.10. *Suppose* $T \geq 3$ *is a large parameter. Then we have the estimate*

$$\sum_{\substack{n,m\leq T \\ n\neq m}} \frac{d_2(n)d_2(m)}{m^{3/4}n^{3/4}|n^{1/2}-m^{1/2}|} \ll \log^4 T.$$

PROOF. See, for example, Ivić and Zhai [7]. □

Lemma 2.11. *If* $\sqrt{n}+\sqrt{m}-\sqrt{k} \neq 0$, *then we have*

$$|\sqrt{n}+\sqrt{m}-\sqrt{k}| \gg \frac{1}{\sqrt{nmk}}.$$

PROOF. This is Lemma 2.5 of the author [18]. □

3. *Proof of Theorem* 1.1

Let $T \geq 10$ be a large parameter. We only need to estimate the integral

$$P(T) := \int_T^{2T} \Delta_3(x)\Delta_2^2(x)\,\mathrm{d}x.$$

Suppose $T^\varepsilon \ll y \ll T^{1/3}$ is a parameter to be determined later. Write

$$P(T) = P_1(T) + P_2(T) \tag{3.1}$$

where

$$P_1(T) := \int_T^{2T} \Delta_{31}(x;y)\Delta_2^2(x)\,\mathrm{d}x,$$

and

$$P_2(T) := \int_T^{2T} \Delta_{32}(x;y)\Delta_2^2(x)\,\mathrm{d}x.$$

By Lemma 2.3 and (1.3) with $A_0 = 4$ we get

$$P_2(T) \ll T^{11/6+\varepsilon}y^{-1/6}. \tag{3.2}$$

It suffices for us to bound $P_1(T)$. Suppose $T^\varepsilon \ll Y \ll T^{1/3}$ is a parameter to be determined later. Write

$$P_1(T) = P_{11}(T) + 2P_{12}(T) + P_{13}(T), \tag{3.3}$$

where

$$P_{11}(T) := \int_T^{2T} \Delta_{31}(x;y)\Delta_{21}^2(x;Y)\,\mathrm{d}x,$$

$$P_{12}(T) := \int_T^{2T} \Delta_{31}(x;y)\Delta_{21}(x;Y)\Delta_{22}(x;Y)\,\mathrm{d}x,$$

$$P_{13}(T) := \int_T^{2T} \Delta_{31}(x;y)\Delta_{22}^2(x;Y)\,\mathrm{d}x.$$

From Lemma 2.2, (2.4) of Lemma 2.4, (2.7) of Lemma 2.5, and using Hölder's inequality, we get

$$\begin{aligned} P_{12}(T) &\ll \left(\int_T^{2T} \Delta_{31}^4(x;y)\,\mathrm{d}x\right)^{1/4}\left(\int_T^{2T} \Delta_{21}^4(x;Y)\,\mathrm{d}x\right)^{1/4}\left(\int_T^{2T} \Delta_{22}^2(x;Y)\,\mathrm{d}x\right)^{1/2} \\ &\ll T^{11/6+\varepsilon}Y^{-1/4}. \end{aligned} \tag{3.4}$$

By Lemma 2.2, (2.8) of Lemma 2.5 and Hölder's inequality we get

$$P_{13}(T) \ll \left(\int_T^{2T} \Delta_{31}^4(x;y)\,\mathrm{d}x\right)^{1/4} \left(\int_T^{2T} 1dt\right)^{1/4} \left(\int_T^{2T} \Delta_{22}^4(x;Y)\,\mathrm{d}x\right)^{1/2}$$
$$\ll T^{11/6+\varepsilon}Y^{-1/6}. \tag{3.5}$$

We now bound $P_{11}(T)$. By the elementary formula

$$\cos\alpha_1\cos\alpha_2\cos\alpha_3 = \frac{1}{4}\sum_{(j_1,j_2)\in\{0,1\}^2} \cos\left[\alpha_1 + (-1)^{j_1}\alpha_2 + (-1)^{j_2}\alpha_3\right]$$

we can write

$$\begin{aligned}
&\Delta_{31}(x;y))(\Delta_{21}(x;Y))^2\\
&\qquad = \frac{x^{5/6}}{2\sqrt{3}\pi^3}\sum_{(j_1,j_2)\in\{0,1\}^2}\sum_{n\le y}\sum_{m_1\le Y}\sum_{m_2\le Y}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}}\\
&\qquad\qquad \times\cos\Big[6\pi(nx)^{1/3} + (-1)^{j_1}4\pi(xm_1)^{1/2}\\
&\qquad\qquad\qquad + (-1)^{j_2}4\pi(xm_2)^{1/2} - \frac{\pi}{4}((-1)^{j_1} + (-1)^{j_2})\Big]\\
&\qquad = S_1(x) + S_2(x) - S_3(x),
\end{aligned}$$

where

$$S_1(x) := \frac{x^{5/6}}{2\sqrt{3}\pi^3}\sum_{n\le y}\sum_{m_1\le Y}\sum_{m_2\le Y}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}} \times\sin\left[6\pi(nx)^{1/3} + 4\pi(xm_1)^{1/2} + 4\pi(xm_2)^{1/2}\right],$$

$$S_2(x) := \frac{x^{5/6}}{2\sqrt{3}\pi^3}\sum_{n\le y}\sum_{m_1\le Y}\sum_{m_2\le Y}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}} \times\sin\left[6\pi(nx)^{1/3} - 4\pi(xm_1)^{1/2} - 4\pi(xm_2)^{1/2}\right],$$

$$S_3(x) := \frac{x^{5/6}}{\sqrt{3}\pi^3}\sum_{n\le y}\sum_{m_1\le Y}\sum_{m_2\le Y}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}} \times\cos\left[6\pi(nx)^{1/3} - 4\pi(xm_1)^{1/2} + 4\pi(xm_2)^{1/2}\right].$$

Let

$$f_1(x) = 6\pi(nx)^{1/3} + 4\pi(xm_1)^{1/2} + 4\pi(xm_2)^{1/2}.$$

Then for any $T \le x \le 2T$, we have

$$f_1'(x) \gg \frac{m_1^{1/2} + m_2^{1/2}}{T^{1/2}}.$$

By Lemma 2.8, Lemma 2.9 and the inequality $a^2 + b^2 \ge 2ab$ we have

$$\begin{aligned}\int_T^{2T} S_1(x)\,\mathrm{d}x &\ll T^{4/3} \sum_{n\le y} \sum_{m_1\le Y} \sum_{m_2\le Y} \frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}} m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} (m_1^{1/2} + m_2^{1/2})} \\ &\ll T^{4/3} \sum_{n\le y} \sum_{m_1\le Y} \sum_{m_2\le Y} \frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}} m_1 m_2} \\ &\ll T^{4/3} y^{1/3} \log^6 T. \end{aligned} \tag{3.6}$$

Let

$$f_2(x) = 6\pi(nx)^{1/3} - 4\pi(xm_1)^{1/2} - 4\pi(xm_2)^{1/2}.$$

Then

$$f_2'(x) = 2\pi \left(\frac{n^{1/3}}{x^{2/3}} - \frac{m_1^{1/2} + m_2^{1/2}}{x^{1/2}} \right).$$

Since $y \ll T^{1/3}$, it is easy to see that for any $n \le y$, $m_1 \le Y$, $m_2 \le Y$, and $T \le x \le 2T$ we have

$$|f_2'(x)| \gg \frac{m_1^{1/2} + m_2^{1/2}}{x^{1/2}}.$$

So similar to $S_1(x)$, by Lemma 2.8 and Lemma 2.9 we have

$$\begin{aligned}\int_T^{2T} S_2(x)\,\mathrm{d}x &\ll T^{4/3} \sum_{n\le y} \sum_{m_1\le Y} \sum_{m_2\le Y} \frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}} m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} (m_1^{1/2} + m_2^{1/2})} \\ &\ll T^{4/3} y^{1/3} \log^6 T. \end{aligned} \tag{3.7}$$

Now we consider the contribution of $S_3(x)$. We write

$$S_3(x) = S_{31}(x) + S_{32}(x) + S_{33}(x), \tag{3.8}$$

where

$$S_{31}(x):=\frac{x^{5/6}}{\sqrt{3}\pi^3}\sum_{n\le y}\sum_{m\le Y}\frac{d_3(n)d_2^2(m)}{n^{\frac{2}{3}}m^{\frac{3}{2}}}\cos\left[6\pi(nx)^{1/3}\right],$$

$$S_{32}(x):=\frac{x^{5/6}}{\sqrt{3}\pi^3}\sum_{n\le y}\sum_{\substack{m_1\le Y,m_2\le Y\\ m_1<m_2}}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}}$$
$$\times\cos\left[6\pi(nx)^{1/3}-4\pi(xm_1)^{1/2}+4\pi(xm_2)^{1/2}\right],$$

$$S_{33}(x):=\frac{x^{5/6}}{\sqrt{3}\pi^3}\sum_{n\le y}\sum_{\substack{m_1\le Y,m_2\le Y\\ m_1>m_2}}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}}$$
$$\times\cos\left[6\pi(nx)^{1/3}-4\pi(xm_1)^{1/2}+4\pi(xm_2)^{1/2}\right].$$

Obviously, by Lemma 2.8 and Lemma 2.9 we have

$$\int_T^{2T} S_{31}(x)\,\mathrm{d}x \ll T^{3/2}\log^3 T. \tag{3.9}$$

Let

$$f_3(x)=6\pi(nx)^{1/3}-4\pi(xm_1)^{1/2}+4\pi(xm_2)^{1/2}.$$

Then

$$f_3'(x)=2\pi\left(\frac{n^{1/3}}{x^{2/3}}-\frac{m_1^{1/2}-m_2^{1/2}}{x^{1/2}}\right).$$

If $m_1<m_2$, it is easy to see that for any $T\le x\le 2T$ we have

$$|f_3'(x)|\gg\frac{|m_2^{1/2}-m_1^{1/2}|}{T^{1/2}}.$$

So by Lemma 2.8 and Lemma 2.10 we have

$$\int_T^{2T} S_{32}(x)\,\mathrm{d}x \ll T^{4/3}\sum_{n\le y}\sum_{m_1\le Y}\sum_{m_2\le Y}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}|m_1^{1/2}-m_2^{1/2}|}$$
$$\ll T^{4/3}y^{1/3}\log^6 T. \tag{3.10}$$

Finally, we consider the contribution of $S_{33}(x)$. By a splitting argument, $S_{33}(x)$ can be written as a sum of $O(\log^3 T)$ terms of the form

$$J(x;N,M_1,M_2) := \frac{x^{5/6}}{\sqrt{3}\pi^3}\sum_{n\sim N}\sum_{\substack{m_1\sim M_1, m_2\sim M_2\\ m_1>m_2}}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}}\cos f_3(x).$$

We write further that

$$J(x;N,M_1,M_2) = J_1(x;N,M_1,M_2) + J_2(x;N,M_1,M_2) + J_3(x;N,M_1,M_2),$$

where

$$\begin{aligned} SC(J_1):\ & n\sim N,\quad m_1\sim M_1,\quad m_2\sim M_2,\\ & m_1>m_2,\quad n^{1/3} > 10T^{1/6}(m_1^{1/2}-m_2^{1/2}),\\ SC(J_2):\ & n\sim N,\quad m_1\sim M_1,\quad m_2\sim M_2,\\ & m_1>m_2,\quad n^{1/3} < \frac{1}{10}T^{1/6}(m_1^{1/2}-m_2^{1/2}),\\ SC(J_3):\ & n\sim N,\quad m_1\sim M_1,\quad m_2\sim M_2,\\ & m_1>m_2,\quad n^{1/3} \asymp T^{1/6}(m_1^{1/2}-m_2^{1/2}). \end{aligned}$$

For J_1, we have $|f_3(x)| \gg n^{1/3}T^{-2/3}$. So by Lemma 2.8 and Lemma 2.9 we get

$$\begin{aligned}\int_T^{2T} J_1(x;N,M_1,M_2)\,\mathrm{d}x &\ll T^{3/2}\sum_{n\le y}\sum_{m_1\le Y}\sum_{m_2\le Y}\frac{d_3(n)d_2(m_1)d_2(m_2)}{nm_1^{\frac{3}{4}}m_2^{\frac{3}{4}}}\\ &\ll T^{3/2}Y^{1/2}\log^6 T. \end{aligned}\tag{3.11}$$

For J_2, we have $|f_3(x)| \gg |m_1^{1/2}-m_2^{1/2}|/T^{1/2}$. By Lemma 2.8 and Lemma 2.10 we get

$$\begin{aligned}\int_T^{2T} J_2(x;N,M_1,M_2)\,\mathrm{d}x &\ll T^{4/3}\sum_{n\le y}\sum_{m_1\le Y}\sum_{m_2\le Y}\frac{d_3(n)d_2(m_1)d_2(m_2)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}\left|m_1^{1/2}-m_2^{1/2}\right|}\\ &\ll T^{4/3}y^{1/3}\log^6 T. \end{aligned}\tag{3.12}$$

Now we consider the contribution of $J_3(x;N,M_1,M_2)$. We only need to bound the integral $\int_T^{2T} x^{5/6}\cos f_3(x)\,\mathrm{d}x$. Suppose $\eta > 0$ is a small parameter to be determined later. Let

$$I_1 = \{T\le x\le 2T : |f_3(x)| > \eta\},\quad I_2 = \{T\le x\le 2T : |f_3(x)|\le\eta\}.$$

Without loss of generality, we suppose that both I_1 and I_2 are not empty.

Suppose $x \in I_2$. Then we have $|n^{1/3} - x^{1/6}(m_1^{1/2} - m_2^{1/2})| \le \eta(2T)^{2/3}$, which implies that

$$x^{1/6} = \frac{n^{1/3}}{m_1^{1/2} - m_2^{1/2}} + O\left(\frac{\eta T^{2/3}}{m_1^{1/2} - m_2^{1/2}}\right).$$

Obviously I_2 is a closed interval. Let $I_2 = [x_1, x_2]$. By the Lagrange theorem, we get

$$x_2 - x_1 \ll \frac{\eta T^{3/2}}{(m_1^{1/2} - m_2^{1/2})},$$

which implies that

$$\int_{I_2} x^{5/6} \cos f_3(x)\,\mathrm{d}x \ll T^{5/6}(x_2 - x_1) \ll \frac{\eta T^{14/6}}{m_1^{1/2} - m_2^{1/2}}.$$

By Lemma 2.8 we have

$$\int_{I_1} x^{5/6} \cos f_3(x)\,\mathrm{d}x \ll \frac{T^{5/6}}{\eta}.$$

So we get by choosing $\eta = |m_1^{1/2} - m_2^{1/2}|^{1/2}/T^{3/4}$ that

$$\int_T^{2T} x^{5/6} \cos f_3(x)\,\mathrm{d}x \ll \frac{T^{19/12}}{(m_1^{1/2} - m_2^{1/2})^{1/2}}.$$

Since $n^{1/3} \asymp T^{1/6}\left|m_1^{1/2} - m_2^{1/2}\right|$, the above bound becomes

$$\int_T^{2T} x^{5/6} \cos f_3(x)\,\mathrm{d}x \ll \frac{T^{5/3}}{n^{1/6}}, \tag{3.13}$$

which implies that

$$\int_T^{2T} J_3(x; N, M_1, M_2)\,\mathrm{d}x \ll T^{5/3} \sum_{n\sim N} \sum_{\substack{m_1\sim M_1, m_2\sim M_2 \\ |m_1^{1/2} - m_2^{1/2}| \asymp \frac{n^{1/3}}{T^{1/6}}}} \frac{d_3(n) d_2(m_1) d_2(m_2)}{n^{\frac{5}{6}} m_1^{\frac{3}{4}} m_2^{\frac{3}{4}}}$$

$$\ll \frac{T^{5/3}}{N^{5/6} M_1^{3/2}} \sum_{n\sim N} d_3(n) \sum_{\substack{m_1\sim M_1, m_2\sim M_2 \\ |m_1^{1/2} - m_2^{1/2}| \asymp \frac{n^{1/3}}{T^{1/6}}}} d_2(m_1) d_2(m_2). \tag{3.14}$$

It suffices for us to bound

$$U = \sum_{n\sim N} d_3(n) \sum_{\substack{m_1\sim M_1, m_2\sim M_2 \\ |m_1^{1/2}-m_2^{1/2}|\asymp \frac{n^{1/3}}{T^{1/6}}}} d_2(m_1)d_2(m_2).$$

It is easy to see that if $|m_1^{1/2} - m_2^{1/2}| \asymp \frac{n^{1/3}}{T^{1/6}}$, then for fixed m_1, the number of m_2's is $\ll 1+\frac{n^{1/3}m_1^{1/2}}{T^{1/6}}$. By the symmetry of m_1 and m_2, Cauchy's inequality and Lemma 2.9 we have

$$\begin{aligned} U &\ll \sum_{n\sim N} d_3(n) \sum_{m_1\sim M_1} d_2^2(m_1) \sum_{\substack{m_2\sim M_2 \\ |m_1^{1/2}-m_2^{1/2}|\asymp \frac{n^{1/3}}{T^{1/6}}}} 1 \\ &\ll \sum_{n\sim N} d_3(n) \sum_{m_1\sim M_1} d_2^2(m_1)\left(1+\frac{n^{1/3}m_1^{1/2}}{T^{1/6}}\right) \\ &\ll NM_1 \log^5 T + \frac{N^{4/3}}{T^{1/6}} M_1^{3/2} \log^5 T. \end{aligned} \tag{3.15}$$

Inserting (3.15) into (3.14) we get

$$\begin{aligned} \int_T^{2T} J_3(x; N, M_1, M_2)\,\mathrm{d}x &\ll \frac{T^{5/3}}{N^{5/6}M_1^{3/2}}\left(NM_1 \log^5 T + \frac{N^{4/3}}{T^{1/6}} M_1^{3/2} \log^5 T\right) \\ &\ll T^{5/3}y^{1/6}\log^5 T + T^{3/2}y^{1/2}\log^5 T. \end{aligned} \tag{3.16}$$

From (3.11), (3.12) and (3.16) we have

$$\int_T^{2T} S_{33}(x)\,\mathrm{d}x \ll K(T,y,Y),$$

which combining with (3.8)-(3.10) gives

$$\int_T^{2T} S_3(x)\,\mathrm{d}x \ll K(T,y,Y), \tag{3.17}$$

where

$$\begin{aligned} K(T,y,Y) := {}& T^{5/3}y^{1/6}\log^9 T + T^{3/2}y^{1/2}\log^9 T \\ &+ T^{4/3}y^{1/3}\log^6 T + T^{3/2}Y^{1/2}\log^4 T. \end{aligned}$$

From (3.6), (3.7) and (3.17) we obtain

$$P_{11}(T) \ll K(T, y, Y). \tag{3.18}$$

From (3.1), (3.2) and (3.18) we get by choosing $y = Y = T^{1/3}$ that

$$P(T) \ll K(T, y, Y) + T^{11/6+\varepsilon} Y^{-1/6} + T^{11/6+\varepsilon} y^{-1/6} \ll T^{16/9+\varepsilon}, \tag{3.19}$$

which combining with a splitting argument proves Theorem 1.1.

4. *Proof of Theorem* 1.2

We shall estimate the integral

$$Q(T) := \int_T^{2T} \Delta_3(x)\Delta_2^3(x)\,\mathrm{d}x.$$

Suppose $T^\varepsilon \ll y \ll T^{1/3}$ is a parameter to be determined later. Write

$$Q(T) = Q_1(T) + Q_2(T), \tag{4.1}$$

where

$$Q_1(T) := \int_T^{2T} \Delta_{31}(x; y)\Delta_2^3(x)\,\mathrm{d}x, \quad Q_2(T) := \int_T^{2T} \Delta_{32}(x; y)\Delta_2^3(x)\,\mathrm{d}x.$$

By Lemma 2.3 and (1.3) with $A_0 = 6$ we have

$$Q_2(T) \ll T^{25/12+\varepsilon} y^{-1/6}. \tag{4.2}$$

It suffices for us to bound $Q_1(T)$. Suppose $T^\varepsilon \ll Y \ll T^{1/3}$ is a parameter to be determined later. We write

$$Q_1(T) = Q_{11}(T) + 3Q_{12}(T) + 3Q_{13}(T) + Q_{14}(T), \tag{4.3}$$

where

$$Q_{11}(T) := \int_T^{2T} \Delta_{31}(x; y)(\Delta_{21}(x; Y))^3\,\mathrm{d}x,$$

$$Q_{12}(T) := \int_T^{2T} \Delta_{31}(x; y)\Delta_{21}^2(x; Y)\Delta_{22}(x; Y)\,\mathrm{d}x,$$

$$Q_{13}(T) := \int_T^{2T} \Delta_{31}(x; y)\Delta_{21}(x; Y)(\Delta_{22}(x; Y))^2\,\mathrm{d}x,$$

$$Q_{14}(T) := \int_T^{2T} \Delta_{31}(x; y)(\Delta_{22}(x; Y))^3\,\mathrm{d}x.$$

By Lemma 2.2, (2.5) of Lemma 2.4, (2.7) of Lemma 2.5 and Hölder's inequality, we have

$$Q_{12}(T) \ll \left(\int_T^{2T} \Delta_{31}^4(x;y)\,\mathrm{d}x\right)^{\frac{1}{4}} \left(\int_T^{2T} \Delta_{21}^8(x;Y)\,\mathrm{d}x\right)^{\frac{1}{4}} \left(\int_T^{2T} \Delta_{22}^2(x;Y)\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\ll T^{25/12+\varepsilon}Y^{-1/4}. \tag{4.4}$$

By Lemma 2.2, (2.4) of Lemma 2.4, (2.8) of Lemma 2.5, as well as Hölder's inequality, we have

$$Q_{13}(T) \ll \left(\int_T^{2T} \Delta_{31}^4(x;y)\,\mathrm{d}x\right)^{\frac{1}{4}} \left(\int_T^{2T} \Delta_{21}^4(x;Y)\,\mathrm{d}x\right)^{\frac{1}{4}} \left(\int_T^{2T} \Delta_{22}^4(x;Y)\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\ll T^{25/12+\varepsilon}Y^{-1/6}. \tag{4.5}$$

By Lemma 2.2, (2.8) of Lemma 2.5 and Hölder's inequality, we have

$$Q_{14}(T) \ll \left(\int_T^{2T} \Delta_{31}^4(x;y)\,\mathrm{d}x\right)^{\frac{1}{4}} \left(\int_T^{2T} \Delta_{22}^4(x;Y)\,\mathrm{d}x\right)^{\frac{3}{4}}$$
$$\ll T^{25/12+\varepsilon}Y^{-1/4}. \tag{4.6}$$

It suffices to bound $Q_1(T)$. By the elementary formula

$$\prod_{l=1}^4 \cos\alpha_l = \frac{1}{8}\sum_{(j_1,j_2,j_3)\in\{0,1\}^3} \cos\left[(-1)^{j_1}\alpha_1 + (-1)^{j_2}\alpha_2 + (-1)^{j_3}\alpha_3 + \alpha_4\right],$$

we have

$$\Delta_{31}(x;y)(\Delta_{21}(x;Y))^3$$
$$= \frac{x^{\frac{13}{12}}}{2\sqrt{6}\,\pi^4}\sum_{n\le y}\sum_{m_1,m_2,m_3\le Y} \frac{d_3(n)d_2(m_1)d_2(m_2)d_2(m_3)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}m_3^{\frac{3}{4}}}\cos(6\pi(xn)^{1/3})$$
$$\times\cos\left[4\pi(xm_1)^{1/2}-\frac{\pi}{4}\right]\cos\left[4\pi(xm_2)^{1/2}-\frac{\pi}{4}\right]\cos\left[4\pi(xm_3)^{1/2}-\frac{\pi}{4}\right]$$
$$= \frac{x^{13/12}}{16\sqrt{6}\,\pi^4}\sum_{(j_1,j_2,j_3)\in\{0,1\}^3}\sum_{n\le y}\sum_{m_1,m_2,m_3\le Y} \frac{d_3(n)d_2(m_1)d_2(m_2)d_2(m_3)}{n^{\frac{2}{3}}m_1^{\frac{3}{4}}m_2^{\frac{3}{4}}m_3^{\frac{3}{4}}}$$
$$\times\cos f(x;n,\mathbf{m},\mathbf{j}), \tag{4.7}$$

where $\mathbf{m} = (m_1, m_2, m_3)$, $\mathbf{j} = (j_1, j_2, j_3)$, and

$$\begin{aligned} f(x; n, \mathbf{m}, \mathbf{j}) = 6\pi(xn)^{1/3} &+ (-1)^{j_1} 4\pi(xm_1)^{1/2} \\ &+ (-1)^{j_2} 4\pi(xm_2)^{1/2} + (-1)^{j_3} 4\pi(xm_3)^{1/2} \\ &+ \frac{\pi}{4}((-1)^{j_1+1} + (-1)^{j_2+1} + (-1)^{j_3+1}). \end{aligned} \tag{4.8}$$

For each $\mathbf{j}$, we shall estimate the integral

$$H(T; \mathbf{j}) := \int_T^{2T} x^{13/12} \sum_{n \le y} \sum_{\mathbf{m}} \frac{d_3(n) d_2(m_1) d_2(m_2) d_2(m_3)}{n^{\frac{2}{3}} m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} m_3^{\frac{3}{4}}} \cos f(x; n, \mathbf{m}, \mathbf{j}) \, \mathrm{d}x.$$

Since $y \ll T^{1/3}$, it is easy to see that if $\mathbf{j} = (0,0,0)$ or $\mathbf{j} = (1,1,1)$, we have

$$|f'(x; n, \mathbf{m}, \mathbf{j})| \gg m_1^{1/2} + m_2^{1/2} + m_3^{1/2}.$$

By Lemma 2.8 and Lemma 2.6 we get

$$\begin{aligned} H(T; \mathbf{j}) &\ll T^{19/12} \sum_{n \le y} \sum_{m_1, m_2, m_3 \le Y} \frac{d_3(n) d_2(m_1) d_2(m_2) d_2(m_3)}{n^{\frac{2}{3}} m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} m_3^{\frac{3}{4}} (m_1^{1/2} + m_2^{1/2} + m_3^{1/2})} \\ &\ll T^{19/12} y^{1/3} Y^{1/4} \log^6 T. \end{aligned} \tag{4.9}$$

Now we suppose $\mathbf{j} \ne (0,0,0)$ and $\mathbf{j} \ne (1,1,1)$. Let

$$\alpha = (-1)^{j_1}(m_1)^{1/2} + (-1)^{j_2}(m_2)^{1/2} + (-1)^{j_3}(m_3)^{1/2}.$$

Then $f'(x; n, \mathbf{m}, \mathbf{j}) = 2\pi \left(n^{1/3}/x^{2/3} + \alpha/x^{1/2} \right)$.

If $n^{1/3} \ge 10 T^{1/6} |\alpha|$, then by Lemma 2.8 we have

$$\int_T^{2T} x^{13/12} \cos f(x; n, \mathbf{m}, \mathbf{j}) \, \mathrm{d}x \ll T^{7/4} n^{-1/3},$$

whose contribution to $H(T; \mathbf{j})$ is

$$\begin{aligned} &\ll T^{7/4} \sum_{n \le y} \sum_{m_1, m_2, m_3 \le Y} \frac{d_3(n) d_2(m_1) d_2(m_2) d_2(m_3)}{n m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} m_3^{\frac{3}{4}}} \\ &\ll T^{7/4} Y^{3/4} \log^6 T. \end{aligned} \tag{4.10}$$

If $n^{1/3} \leq \frac{1}{10}T^{1/6}|\alpha|$, then by Lemma 2.8 again we have

$$\int_T^{2T} x^{13/12} \cos f(x; n, \mathbf{m}, \mathbf{j})\, \mathrm{d}x \ll T^{19/12}/|\alpha|.$$

whose contribution to $H(T; \mathbf{j})$ is (via Lemma 2.6)

$$\begin{aligned} &\ll T^{19/12} \sum_{n \leq y} \sum_{m_1, m_2, m_3 \leq Y} \frac{d_3(n) d_2(m_1) d_2(m_2) d_2(m_3)}{n^{2/3} m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} m_3^{\frac{3}{4}} |\alpha|} \\ &\ll T^{19/12+\varepsilon} y^{1/3} Y^{/4}. \end{aligned} \tag{4.11}$$

Now suppose $n^{1/3} \asymp T^{1/6}|\alpha|$, then similarly to (3.13) we have the estimate

$$\int_T^{2T} x^{\frac{13}{12}} \cos f(x; n, \mathbf{m}, \mathbf{j})\, \mathrm{d}x \ll \frac{T^{23/12}}{|\alpha|^{1/6}},$$

whose contribution of $H(T; \mathbf{j})$ is

$$\ll T^{23/12} \sum_{n \leq y} \frac{d_3(n)}{n^{5/6}} G(T, n), \tag{4.12}$$

where

$$G(T, n) := \sum_{\substack{m_1, m_2, m_3 \leq Y \\ |\alpha| \leq n^{1/3} T^{-1/6}}} \frac{d_2(m_1) d_2(m_2) d_2(m_3)}{m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} m_3^{\frac{3}{4}}}.$$

By a splitting argument and Lemma 2.7 we get (suppose $M_1 \ll M_2 \ll M_3 \ll Y$)

$$\begin{aligned} G(T, n) &\ll \log^3 Y \sum_{\substack{m_1 \sim M_1, m_2 \sim M_2, m_3 \sim M_3 \\ |\alpha| \leq n^{1/3} T^{-1/6}}} \frac{d_2(m_1) d_2(m_2) d_2(m_3)}{m_1^{\frac{3}{4}} m_2^{\frac{3}{4}} m_3^{\frac{3}{4}}} \\ &\ll \frac{Y^\varepsilon}{M_1^{\frac{3}{4}} M_2^{\frac{3}{4}} M_3^{\frac{3}{4}}} \times \left(\frac{n^{1/3}}{T^{1/6}} M_1 M_2 M_3^{1/2} + M_1^{1/2} M_2^{1/2} \right) \\ &\ll \frac{n^{1/3} Y^{1/4+\varepsilon}}{T^{1/6}} + \frac{Y^\varepsilon}{M_1^{\frac{1}{4}} M_2^{\frac{1}{4}} M_3^{\frac{3}{4}}} \\ &\ll \frac{n^{1/3} Y^{1/4+\varepsilon}}{T^{1/6}} + \frac{Y^\varepsilon}{(M_1 M_2 M_3)^{5/12}}. \end{aligned}$$

By Lemma 2.11 we have $|\alpha| \gg 1/\sqrt{M_1 M_2 M_3}$, which combining with $|\alpha| \leq n^{1/3}T^{-1/6}$ implies that

$$\frac{1}{(M_1 M_2 M_3)^{5/12}} \ll \frac{n^{5/18}}{T^{5/36}}.$$

So we get

$$G(T,n) \ll \frac{n^{1/3}Y^{1/4+\varepsilon}}{T^{1/6}} + \frac{n^{5/18}Y^{\varepsilon}}{T^{5/36}}.$$

Inserting the above bound into (4.12) we get via Lemma 2.9 that the contribution of the case $n^{1/3} \asymp T^{1/6}|\alpha|$ is

$$\ll T^{7/4+\varepsilon}y^{1/2}Y^{1/4} + T^{16/9+\varepsilon}y^{4/9}. \tag{4.13}$$

From (4.10), (4.11) and (4.13) we see that if $\mathbf{j} \neq (0,0,0)$ and $\mathbf{j} \neq (1,1,1)$, then we have

$$H(T;\mathbf{j}) \ll L(T,y,Y)T^{\varepsilon} \tag{4.14}$$

by noting that $T^{19/12}y^{1/3}T^{1/4} \ll T^{7/4}y^{1/2}Y^{1/4}$, where

$$L(T,y,Y) := T^{7/4}y^{1/2}Y^{1/4} + T^{16/9}y^{4/9} + T^{7/4}Y^{3/4}.$$

From (4.7), (4.9) and (4.14) we get

$$Q_{11}(T) \ll L(T,y,Y)T^{\varepsilon},$$

which combining with (4.4)–(4.6) gives

$$Q_1(T) \ll L(T,y,Y)T^{\varepsilon} + T^{25/12+\varepsilon}Y^{-1/6}. \tag{4.15}$$

From (4.1), (4.2) and (4.15) we get by choosing $y = Y = T^{1/3}$ that

$$\begin{aligned} Q(T) &\ll L(T,y,Y)T^{\varepsilon} + T^{25/12+\varepsilon}Y^{-1/6} + T^{25/12+\varepsilon}y^{-1/6} \\ &\ll T^{73/36+\varepsilon}, \end{aligned}$$

which combining with a splitting argument proves Theorem 1.2.

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