ISSN 0561-7332

ACADEMIE SERBE DES SCIENCES ET DES ARTS

BULLETIN

TOME CLII

CLASSE DES SCIENCES MATHEMATIQUES ET NATURELLES

SCIENCES MATHEMATIQUES

N° 44

BEOGRAD 2019

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R é d a c t e u r

GRADIMIR V. MILOVANOVIĆ
Membre de l'Académie

B E O G R A D 2 0 1 9

Publie et impimé par
Académie serbe des sciences et des arts
Beograd, Knez Mihailova 35

Tirage 300 exemplaires

TABLE DES MATIÈRES

Bulletin T. CLII de l'Académie serbe des sciences et des arts − 2019
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 44

ON CERTAIN SUMS OVER ORDINATES OF ZETA ZEROS III

ALEKSANDAR IVIĆ

Dedicated to the memory of Professor Bogoljub Stanković (1924–2018)

(Presented at the 7th Meeting, held on October 25, 2019)

A b s t r a c t. The upper bound

$$\int_2^T |G(\tfrac{1}{2}+\mathrm{i}t)|^2 \,\mathrm{d}t \ll T\log^2 T$$

is proved, where initially $G(s) = \sum_{\gamma>0} \gamma^{-s}$*. Here* γ *denotes ordinates of complex zeros of the Riemann zeta-function* $\zeta(s)$*. This coincides with the lower bound for the integral in question.*

AMS Mathematics Subject Classification (2010): 11M06
Key Words: Riemann zeta-function, sums over ordinates, mean square estimates.

1. *Introduction*

This paper is a continuation of the author's work [5] and the joint work [1]. It deals with a mean square estimate for the function

$$G(s) := \sum_{\gamma>0} \gamma^{-s} \qquad (s = \sigma + \mathrm{i}t; \sigma, t \in \mathbb{R}, \sigma > 1),$$

where γ denotes ordinates of complex zeros of the Riemann zeta-function $\zeta(s)$. Here, as usual, the zeros are counted with their respective multiplicities. For a comprehensive account on $\zeta(s)$, the reader is referred to the monographs of E.C. Titchmarsh [9] and the author [4]. The series for $G(s)$ does not converge for $\operatorname{Re} s \leqslant 1$, but the function itself possesses unconditionally analytic continuation at least to the region $\operatorname{Re} s > -1$. The mean square estimate

$$T\log^2 T \ll \int_0^T \left|G\left(\tfrac{1}{2}+\mathrm{i}t\right)\right|^2 \,\mathrm{d}t \ll T\log^2 T\sqrt{\log\log T} \tag{1.1}$$

was proved in [1]. The lower bound in (1.1) is new, and the upper bound improves and rectifies the corresponding result of [5], whose proof was not complete. The Vinogradov symbol $f(x) \ll g(x)$ (same as $f(x) = O(g(x))$) is defined in the usual way: $f(x) \ll g(x)$ means that $|f(x)| \leqslant Cg(x)$ for $x \geqslant x_0$, some constant $C > 0$, provided that $g(x) > 0$ for $x \geqslant x_0$.

The aim of this note is to improve the upper bound in (1.1). Efforts have been made to keep the exposition as complete as possible. We shall prove

Theorem 1.1. *We have*

$$\int_0^T \left|G\left(\tfrac{1}{2}+\mathrm{i}t\right)\right|^2 \,\mathrm{d}t \ll T\log^2 T. \tag{1.2}$$

Remark 1.1. The lower bound in (1.1) and the upper bound in (1.2) are both of the form $T\log^2 T$, so it is plausible to conjecture that

$$\int_0^T \left|G\left(\tfrac{1}{2}+\mathrm{i}t\right)\right|^2 \,\mathrm{d}t = (C+o(1))T\log^2 T \qquad (T\to\infty) \tag{1.3}$$

for some positive constant C. Proving (1.3), however, is out of reach at present.

2. *Proof of Theorem 1.1*

Instead of (1.2) it is sufficient to prove

$$I(T) := \int_{T/2}^T \left|G\left(\tfrac{1}{2}+\mathrm{i}t\right)\right|^2 \,\mathrm{d}t \ll T\log^2 T, \tag{2.1}$$

replace T by $T2^{-j}$ and sum the resulting expressions over $O(\log T)$ values $j = 1, 2, \ldots$.

To start with a workable expression for $G(\frac{1}{2}+\mathrm{i}t)$ we proceed as in [5], using the zero counting function

$$N(T) := \sum_{0<\gamma\leqslant T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + f(T), \tag{2.2}$$

$$f(T) \ll \frac{1}{T}, \quad f'(T) \ll \frac{1}{T^2}, \tag{2.3}$$

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+\mathrm{i}T\right) = \frac{1}{\pi}\operatorname{Im}\left\{\log\zeta\left(\tfrac{1}{2}+\mathrm{i}T\right)\right\} \ll \log T. \tag{2.4}$$

This is known as the Riemann – von Mangoldt formula (see [9] or [4]). Here the argument of $\zeta(\frac{1}{2}+\mathrm{i}T)$ is obtained by continuous variation along the straight lines joining the points 2, $2+\mathrm{i}T$, $\frac{1}{2}+\mathrm{i}T$, starting with the value 0. If T is an ordinate of a zero, then we set $S(T) = S(T+0)$.

Let X be a parameter, to be chosen later, which satisfies $1 \ll X \leqslant T$. Then we write

$$G(s) = \sum_{\gamma\leqslant X}\gamma^{-s} + R(s),$$

say, where on using (2.2) it follows that

$$\begin{aligned} R(s) &= \sum_{\gamma>X}\gamma^{-s} = \int_X^\infty x^{-s}\,\mathrm{d}N(x) \\ &= \int_X^\infty \frac{x^{-s}}{2\pi}\log\frac{x}{2\pi}\,\mathrm{d}x + \int_X^\infty x^{-s}\,\mathrm{d}\big(S(x)+f(x)\big). \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} R(s) &= \frac{X^{1-s}}{2\pi(s-1)}\log\frac{X}{2\pi} + \frac{X^{1-s}}{2\pi(s-1)^2} - X^{-s}\big(S(X)+f(X)\big) \\ &\quad + s\int_X^\infty x^{-s-1}(S(x)+f(x))\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

Initially (2.5) is valid for for $\sigma > 1$, but it possesses meromorphic continuation for all $\sigma > 0$, since $S(x) \ll \log x$. Henceforth we set $s = \frac{1}{2}+\mathrm{i}t$ and choose

$$X = \frac{T}{\log T}. \tag{2.6}$$

Then the contribution of the terms in the first line of (2.5) to $I(T)$ in (2.1) is

$$\ll X\log^2 X\int_{T/2}^{T} t^{-2}\,\mathrm{d}t + \frac{T}{X}\log^2 X \ll \log^3 T. \tag{2.7}$$

Now we use Lemma 4 of the author's paper [6], which says that

$$\int_0^T \left|\int_a^b g(x)x^{-s}\,\mathrm{d}x\right|^2 \mathrm{d}t \leqslant 2\pi\int_a^b g^2(x)x^{1-2\sigma}\,\mathrm{d}x \quad (s=\sigma+\mathrm{i}t, T>0, a<b),$$

if $g(x)$ is a real-valued, integrable function on $[a,b]$, a subinterval of $[2,\infty)$, which is not necessarily finite. With $s=\frac12+\mathrm{i}t, A=X, b=+\infty$ this gives

$$\begin{aligned}\int_{T/2}^{T} &\left|s\int_X^\infty x^{-s-1}\big(S(x)+f(x)\big)\,\mathrm{d}x\right|^2 \mathrm{d}t\\ &\ll T^2\int_X^\infty \big(S^2(x)+f^2(x)\big)x^{-2}\,\mathrm{d}x\\ &\ll T^2X^{-1}\log\log X \ll T\log T\log\log T.\end{aligned} \tag{2.8}$$

Here we used the elementary bound

$$\int_1^X S^2(x)\,\mathrm{d}x \;\ll\; X\log\log X.$$

An elementary calculation shows that

$$\int_{-1}^{1}(1-|y|)\mathrm{e}^{-2\pi\mathrm{i}xy}\,\mathrm{d}y = \Big(\frac{\sin\pi x}{\pi x}\Big)^2.$$

Therefore on applying the Fourier inversion one has

$$\frac12\int_{-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}xy}\left(\frac{\sin\pi x}{\pi x}\right)^2\mathrm{d}x = \begin{cases} 1-|y|, & \text{if } |y|\leqslant 1,\\ 0, & \text{if } |y|>1.\end{cases} \tag{2.9}$$

To estimate the contribution of $\sum_{0<\gamma\leqslant X}\gamma^{-s}$ to $I(T)$ in (2.1) we use (2.9) and the fact that

$$1 \;\leqslant\; \frac{\pi^2}{4}\left(\frac{\sin\frac{\pi t}{2T}}{\frac{\pi t}{2T}}\right)^2 \qquad (|t|\leqslant T).$$

We obtain

$$\int_{T/2}^{T}\Big|\sum_{0<\gamma\leqslant X}\gamma^{-1/2-\mathrm{i}t}\Big|^2\,\mathrm{d}t \ll \int_{T/2}^{T}\left(\frac{\sin\frac{\pi t}{2T}}{\frac{\pi t}{2T}}\right)^2\Big|\sum_{0<\gamma\leqslant X}\gamma^{-1/2-\mathrm{i}t}\Big|^2\,\mathrm{d}t$$

$$\leqslant \sum_{0<\gamma,\gamma'\leqslant X}(\gamma\gamma')^{-1/2}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\pi t}{2T}}{\frac{\pi t}{2T}}\right)^2 \mathrm{e}^{\mathrm{i}t\log\gamma/\gamma'}\,\mathrm{d}t,$$

where both γ and γ' denote ordinates of zeta-zeros, counted with their respective multiplicities. In the last integral we make the change of variable $t = 2Tx$ and apply (2.9) with

$$y \;=\; \frac{T}{\pi}\log\frac{\gamma}{\gamma'}$$

to obtain

$$\int_{T/2}^{T}\Big|\sum_{0<\gamma\leqslant X}\gamma^{-1/2-\mathrm{i}t}\Big|^2\,\mathrm{d}t \ll T\sum_{0<\gamma,\gamma'\leqslant X,|\frac{T}{\pi}\log\frac{\gamma}{\gamma'}|\leqslant 1}(\gamma\gamma')^{-1/2} = T\sum(T), \quad (2.10)$$

say. By symmetry, the portions of $\sum(T)$ in which $\gamma > \gamma'$ and $\gamma < \gamma'$ are equal. Thus we have to distinguish only the cases $\gamma' > \gamma$ and $\gamma' = \gamma$. In the latter case we have a contribution which is

$$\sum_{0<\gamma\leqslant X}\frac{m(\beta+\mathrm{i}\gamma)}{\gamma}, \quad (2.11)$$

where $m(\rho)$ denotes the multiplicity of the zeta-zero $\rho = \beta + \mathrm{i}\gamma$. Let

$$N^*(T) := \sum_{0<\gamma\leqslant T} m(\beta+\mathrm{i}\gamma).$$

Then if we can show that

$$N^*(T) \;\ll\; N(T), \quad (2.12)$$

by partial summation and (2.12) it easily follows that the sum in (2.11) is $\ll \log^2 T$, which suffices for (2.1). But A. Fujii (Theorem 3 of [2]) has shown that

$$N_j(T) \leqslant CN(T)\mathrm{e}^{-Aj} \qquad (A, C > 0, j \geqslant j_0), \quad (2.13)$$

where

$$N_j(T) := \sum_{0<\gamma\leqslant T,m(\beta+\mathrm{i}\gamma)=j} 1.$$

M.A. Korolev [7] later found explicit values of A and C in (2.13). Using (2.13) one has

$$N^*(T) = O(N(T)) + \sum_{j=j_0}^{O(\log T)} jN_j(T) \ll N(T) + N(T)\sum_{j=1}^{\infty} j\mathrm{e}^{-Aj} \ll N(T),$$

since the above series is clearly convergent.

It remains to deal with the case when $\gamma' > \gamma$ in $\sum(T)$ in (2.10). If $\gamma' > \gamma$, then the condition

$$\frac{T}{\pi}\log\frac{\gamma'}{\gamma} \leqslant 1$$

implies, for $T \geqslant T_0$,

$$\gamma < \gamma' \leqslant \mathrm{e}^{\pi/T}\gamma \leqslant \Big(1 + \frac{2\pi}{T}\Big)\gamma \leqslant \gamma + \frac{2\pi}{\log T},$$

in view of (2.6). It transpires that $\gamma' \sim \gamma$ and using (2.2)–(2.4)we have

$$\begin{aligned}\sum(T) &\ll \sum_{0<\gamma\leqslant X, \gamma<\gamma'\leqslant\gamma+(2\pi)/\log T} \frac{1}{\gamma} \qquad (2.14)\\ &= \sum_{0<\gamma\leqslant X} \frac{1}{\gamma}\Big(N(\gamma + \frac{2\pi}{\log T}) - N(\gamma)\Big)\\ &\ll \sum_{0<\gamma\leqslant X} \frac{1}{\gamma}\Big(1 + S(\gamma + \frac{2\pi}{\log T}) - S(\gamma)\Big).\end{aligned}$$

To bound the last sum in (2.14) we invoke the estimate

$$\sum_{0<\gamma\leqslant Q, \gamma+a>0} S(\gamma + a) \ll Q\log Q \qquad (0 \leqslant |a| \leqslant Q, a \in \mathbb{R}) \qquad (2.15)$$

of A. Fujii [3]. Hence, by partial summation, (2.15) yields

$$\sum(T) \ll \log^2 X + \frac{1}{X}X\log X + \int_1^X \frac{x\log x}{x^2}\,\mathrm{d}x \ll \log^2 T. \qquad (2.16)$$

Inserting (2.16) in (2.10) we complete the proof of Theorem 1.1.

Concerning (2.15) Fujii even conjectures that, for any given $\alpha > 0$ and $T \to \infty$ one has

$$\sum_{0<\gamma\leqslant T} S\left(\gamma - \frac{2\pi\alpha}{\log T/(2\pi)}\right) = \frac{T}{2\pi}\left\{\int_0^\alpha \Big(\frac{\sin\pi t}{\pi t}\Big)^2 \mathrm{d}t + o(1)\right\}.$$

This is closely related to H.L. Montgomery's pair correlation conjecture [8] for the distribution of the zeros of $\zeta(s)$.

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Serbian Academy of Sciences and Arts
Knez Mihailova 35
11000 Beograd, Serbia
e-mails: aleksandar.ivic@rgf.bg.ac.rs
aivic_2000@yahoo.com