ISSN 0561-7332

ACADEMIE SERBE DES SCIENCES ET DES ARTS

BULLETIN

TOME CL

CLASSE DES SCIENCES
MATHEMATIQUES ET NATURELLES

SCIENCES MATHEMATIQUES

N° 42

BEOGRAD 2017

BULLETIN

TOME CL

CLASSE DES SCIENCES
MATHEMATIQUES ET NATURELLES

SCIENCES MATHEMATIQUES

N° 42

ISSN 0561-7332

ACADEMIE SERBE DES SCIENCES ET DES ARTS

BULLETIN

TOME CL

CLASSE DES SCIENCES
MATHEMATIQUES ET NATURELLES

SCIENCES MATHEMATIQUES

N° 42

Rédacteur

GRADIMIR V. MILOVANOVIĆ
Membre de l’Académie

BEOGRAD 2017

Publie et impimé par
Académie serbe des sciences et des arts
Beograd, Knez Mihailova 35

Tirage 300 exemplaires

TABLE DES MATIÈRES

Bulletin T. CL de l'Académie serbe des sciences et des arts − 2017
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 42

APPROXIMATION AND CONVERGENCE OF DEGENERATE (a,k)-REGULARIZED C-RESOLVENT FAMILIES

MARKO KOSTIĆ

(Presented at the 6th Meeting, held on September 29, 2017)

A b s t r a c t. The main aim of this paper is to investigate approximation and convergence of degenerate (a,k)-regularized C-resolvent families in locally convex spaces. We continue our previous research of non-degenerate case [*Numer. Funct. Anal. Appl.* **35** (2014), 1579 − 1606]*, following the multivalued linear operators approach to abstract degenerate differential equations. We also consider Laguerre expansions of degenerate (a,k)-regularized C-resolvent families.*

AMS Mathematics Subject Classification (2000): 4A45, 45D05, 47D60, 47D62, 47D99.
Key Words: Degenerate (a,k)-regularized C-resolvent families, approximations, convergence, multivalued linear operators, locally convex spaces.

1. *Introduction and preliminaries*

As mentioned in the abstract, the main aim of this paper is to prove some Trotter-Kato-type formulae for degenerate (a,k)-regularized C-resolvent families in locally convex spaces (cf. [4], [6]–[7], [9], [18] and [21]–[22] for the basic references on

* This research was supported by grant 174024 of Ministry of Science and Technological Development, Republic of Serbia.

approximation of abstract degenerate differential equations), as well as to investigate the Laguerre expansions of degenerate (a,k)-regularized C-resolvent families (cf. [1] for the study of Laguerre expansions of non-degenerate strongly continuous semigroups in Banach spaces). In the second section of paper, we repeat some known facts and definitions from the theory of multivalued linear operators and remind ourselves of the notion of an (a,k)-regularized C-resolvent family subgenerated by a multivalued linear operator ([15]). In the third section of paper, we reconsider our previously established results from [12] in this new setting (the proofs of our structural results are very similar to those of [12] and therefore omitted), while in the fourth section of paper we take up the study of Laguerre expansions of degenerate (a,k)-regularized C-resolvent families.

We use the standard notation throughout the paper. By E we denote a Hausdorff sequentially complete locally convex space, SCLCS for short; the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E. Unless stated otherwise, the seminorms belonging $\circledast$ will be denoted by p, q, r, ... By $L(E)$ we denote the space of all continuous linear mappings from E into E. Let $\mathcal{B}$ be the family of bounded subsets of E and let $p_B(T) := \sup_{x\in B} p(Tx)$, $p \in \circledast$, $B \in \mathcal{B}$, $T \in L(E)$. Then $p_B(\cdot)$ is a seminorm on $L(E)$ and the system $(p_B)_{(p,B)\in\circledast\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E)$. Recall that $L(E)$ is sequentially complete provided that E is barreled. Henceforth A denotes a closed single-valued linear operator acting on E and $C \in L(E)$ denotes an injective operator which satisfies $CA \subseteq AC$. The domain and range of A are denoted by $D(A)$ and $R(A)$, respectively. If $(A_l)_{l\in\mathbb{N}_0}$ $(((R_l(t))_{t\geq 0})_{l\in\mathbb{N}_0})$ is a sequence of closed linear operators on E (strongly continuous operator families in $L(E)$), then we also write A $((R(t))_{t\geq 0})$ in place of A_0 $((R_0(t))_{t\geq 0})$. We refer the reader to [10] for the basic material concerning integration in SCLCSs and vector-valued analytic functions.

The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. The convolution like mapping $*$ is given by

$$f * g(t) := \int_0^t f(t-s)g(s)\,\mathrm{d}s.$$

Set $0^\alpha := 0$ and $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ $(\alpha > 0,\ t > 0)$. For every $n \in \mathbb{N}$, put $\mathbb{N}_n := \{1,\ldots,n\}$.

We use repeatedly the following condition for the function $k(t)$:

(P1) $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ so that

$$\tilde{k}(\lambda) = \mathcal{L}(k)(\lambda) := \lim_{b\to\infty}\int_0^b \mathrm{e}^{-\lambda t}k(t)\,\mathrm{d}t := \int_0^\infty \mathrm{e}^{-\lambda t}k(t)\,\mathrm{d}t$$

exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \beta$. Put $\operatorname{abs}(k) := \inf\{\operatorname{Re} \lambda : \tilde{k}(\lambda) \text{ exists}\}$.

For more details about the Laplace transform of functions with values in Banach and sequentially complete locally convex spaces, the reader may consult [3], [10] and [15].

2. *Multivalued linear operators in locally convex spaces*

In this section, we will present some necessary definitions from the theory of multivalued linear operators. For more details about this topic, we refer the reader to the monographs by R. Cross [8] and A. Favini-A. Yagi [9].

A multivalued map $\mathcal{A} : E \to P(E)$ is said to be a multivalued linear operator (MLO) in E, (MLO) for short, iff the following holds:

(i) $D(\mathcal{A}) := \{x \in E : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of E;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

It is well known that $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$ holds for every $x,\ y \in D(\mathcal{A})$ and for every $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear manifold in E and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Define $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. Then the set $N(\mathcal{A}) := \mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It can be easily seen that $\mathcal{A}^{-1}$ is an MLO in E, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective. If $\mathcal{A},\ \mathcal{B} : E \to P(E)$ are two MLOs, then we define its sum $\mathcal{A} + \mathcal{B}$ by $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. It is clear that $\mathcal{A} + \mathcal{B}$ is likewise an MLO. We write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

Let $\mathcal{A} : E \to P(E)$ and $\mathcal{B} : E \to P(E)$ be two MLOs. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A} : E \to P(E)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : E \to P(E)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A} : E \to P(E)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z,\ \omega \in \mathbb{C}$.

The integer powers of an MLO $\mathcal{A} : E \to P(E)$ are defined recursively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set

$$D(\mathcal{A}^n) := \bigl\{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\bigr\},$$

and

$$\mathcal{A}^n x := \bigl(\mathcal{A}\mathcal{A}^{n-1}\bigr)x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

We say that an MLO $\mathcal{A} : E \to P(E)$ is closed iff for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in E such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that the suppositions $\lim_{\tau\to\infty} x_\tau = x$ and $\lim_{\tau\to\infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Our standing assumptions henceforth will be that $\mathcal{A}$ is an MLO in E, as well as that $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;

(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued bounded operator on E.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$).

We need the following lemma from [15].

Lemma 2.1. *We have*

$$\big(\lambda - \mathcal{A}\big)^{-1}C\mathcal{A} \subseteq \lambda\big(\lambda - \mathcal{A}\big)^{-1}C - C \subseteq \mathcal{A}\big(\lambda - \mathcal{A}\big)^{-1}C, \quad \lambda \in \rho_C(\mathcal{A}).$$

The operator $(\lambda - \mathcal{A})^{-1}C\mathcal{A}$ *is single-valued on* $D(\mathcal{A})$ *and*

$$(\lambda - \mathcal{A})^{-1}C\mathcal{A}x = (\lambda - \mathcal{A})^{-1}Cy,$$

whenever $y \in \mathcal{A}x$ *and* $\lambda \in \rho_C(\mathcal{A})$.

Suppose that Ω is a locally compact, separable metric space, and μ is a locally finite Borel measure defined on Ω. Let $\mathcal{A}$ be a closed MLO, let $f : \Omega \to E$ and $g : \Omega \to E$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then we know that $\int_\Omega f\, d\mu \in D(\mathcal{A})$ and $\int_\Omega g\, d\mu \in \mathcal{A}\int_\Omega f\, d\mu$. In the remaining part of paper, Ω will always be an appropriate subspace of $\mathbb{R}$ and μ will always be the Lebesgue measure defined on Ω.

Definition 2.1. Suppose that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : E \to P(E)$ is an MLO, $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(E)$ is an (a,k)*-regularized* C*-resolvent family* with a *subgenerator* $\mathcal{A}$ iff $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ ($t \in [0,\tau)$), as well as

$$\int_0^t a(t-s)R(s)y\,\mathrm{d}s = R(t)x - k(t)Cx, \text{ whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}.$$

We will occasionally use the following condition:

$$\Bigg(\int\limits_0^t a(t-s)R(s)x\,\mathrm{d}s,\,R(t)x-k(t)Cx\Bigg)\in\mathcal{A},\ t\in[0,\tau),\ x\in E. \tag{2.1}$$

Applying the Laplace transform, we can prove the following result ([15]).

Theorem 2.1. *Let $(R(t))_{t\geq 0}\subseteq L(E)$ be a strongly continuous operator family such that there exists $\omega\geq 0$ satisfying that the family $\{\mathrm{e}^{-\omega t}R(t):t\geq 0\}$ is equicontinuous, and let $\omega_0>\max(\omega,\mathrm{abs}(|a|),\mathrm{abs}(k))$. Suppose that $\mathcal{A}$ is a closed* MLO *in E and $C\mathcal{A}\subseteq\mathcal{A}C$.*

(i) *Assume that $\mathcal{A}$ is a subgenerator of the global (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying (2.1) for all $x=y\in E$. Then, for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_0$ and $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$, the operator $I-\tilde{a}(\lambda)\mathcal{A}$ is injective, $R(C)\subseteq R(I-\tilde{a}(\lambda)\mathcal{A})$,*

$$\tilde{k}(\lambda)\big(I-\tilde{a}(\lambda)\mathcal{A}\big)^{-1}Cx=\int\limits_0^\infty \mathrm{e}^{-\lambda t}R(t)x\,\mathrm{d}t, \tag{2.2}$$

$$(x\in E,\ \ \operatorname{Re}\lambda>\omega_0,\ \ \tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0),$$

$$\left\{\frac{1}{\tilde{a}(\lambda)}:\operatorname{Re}\lambda>\omega_0,\ \tilde{k}(\lambda)\tilde{a}(\lambda)\neq 0\right\}\subseteq\rho_C(\mathcal{A}) \tag{2.3}$$

and $R(s)R(t)=R(t)R(s),\ t,\ s\geq 0$.

(ii) *Assume $(2.2)-(2.3)$. Then $\mathcal{A}$ is a subgenerator of the global (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying (2.1) for all $x=y\in E$ and*

$$R(s)R(t)=R(t)R(s),\quad t,s\geq 0.$$

We refer the reader to [15] for the notion of an exponentially equicontinuous, analytic (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ subgenerated by a multivalued linear operator.

3. *Approximation and convergence of (a,k)-regularized C-resolvent families subgenerated by multivalued linear operators*

The following important result on approximation of abstract vector-valued Laplace transform in locally convex spaces will be frequently used in this paper (cf. [13, Theorem 2.10]).

Theorem 3.1. *Let $f_n \in C([0,\infty) : E)$, $n \in \mathbb{N}$, let the set $\{e^{-\omega t} f_n(t) : n \in \mathbb{N},\ t \geq 0\}$ be bounded for some $\omega \in \mathbb{R}$ and let $\lambda_0 \geq \omega$. Then the following assertions are equivalent:*

(i) *The sequence $(\tilde{f}_n)$ converges pointwise on (λ_0, ∞) and the sequence (f_n) is equicontinuous at each point $t \geq 0$.*

(ii) *The sequence (f_n) converges uniformly on compact subsets of $[0,\infty)$.*

If (ii) *holds and $\lim_{n\to\infty} f_n(t) = f(t)$, $t \geq 0$, then one has $\lim_{n\to\infty} \tilde{f}_n(\lambda) = \tilde{f}(\lambda)$, $\lambda > \lambda_0$.*

Making use of Theorem 2.1 and Theorem 3.1, we can simply prove an extension of [12, Theorem 2.3] for (a,k)-regularized C-resolvent families subgenerated by multivalued linear operators:

Theorem 3.2. *Assume that, for every $n \in \mathbb{N}_0$, $|a_n|(t)$ and $k_n(t)$ satisfy* (P1) *and that $\mathcal{A}_n$ is a closed subgenerator of an (a_n,k_n)-regularized C_n-resolvent family $(R_n(t))_{t\geq 0}$ which satisfies* (2.1) *with $a(t)$, $R(t)$ and $k(t)$ replaced respectively by $a_n(t)$, $R_n(t)$ and $k_n(t)$ ($n \in \mathbb{N}_0$). Assume further that there exists a number $\omega \geq \sup_{n\in\mathbb{N}_0} \max(0, \mathrm{abs}(|a_n|), \mathrm{abs}(k_n))$ such that, for every $p \in \circledast$, there exist $c_p > 0$ and $r_p \in \circledast$ with*

$$p\big(e^{-\omega t} R_n(t)x\big) \leq c_p r_p(x), \quad t \geq 0,\ x \in E,\ n \in \mathbb{N}_0. \tag{3.1}$$

Let $\lambda_0 \geq \omega$. Put $\mathfrak{T} := \{\lambda > \lambda_0 : \widetilde{a_n}(\lambda)\widetilde{k_n}(\lambda) \neq 0 \text{ for all } n \in \mathbb{N}_0\}$. Then the following assertions are equivalent:

(i) $\lim_{n\to\infty} \widetilde{k_n}(\lambda)(I - \widetilde{a_n}(\lambda)\mathcal{A}_n)^{-1} C_n x = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1} C x, \quad \lambda \in \mathfrak{T},\ x \in E$ *and the sequence $(R_n(t)x)_n$ is equicontinuous at each point $t \geq 0$ ($x \in E$).*

(ii) $\lim_{n\to\infty} R_n(t)x = R(t)x,\ t \geq 0,\ x \in E$, *uniformly on compacts of* $[0,\infty)$.

Keeping in mind Lemma 2.1 and Theorem 2.3, it is almost straightforward to formulate an extension of [12, Theorem 2.4] for (a,k)-regularized C-resolvent families subgenerated by multivalued linear operators, as well; the only thing worth noting is that, in the proof of last mentioned theorem, we can replace the sequence $(A_n H_n(\lambda')x)_{n\in\mathbb{N}}$ with the sequence

$$(\widetilde{k_n}(\lambda')\widetilde{a_n}(\lambda')^{-1}[\widetilde{a_n}(\lambda')^{-1}(\widetilde{a_n}(\lambda')^{-1} - \mathcal{A}_n)^{-1} C_n x - C_n x])_{n\in\mathbb{N}}.$$

Theorem 3.3. *Assume that, for every $n \in \mathbb{N}_0$, $|a_n|(t)$ and $k_n(t)$ satisfy* (P1) *and that $\mathcal{A}_n$ is a closed subgenerator of an (a_n,k_n)-regularized C_n-resolvent family*

$(R_n(t))_{t\geq 0}$ which satisfies (2.1) with $a(t)$, $R(t)$ and $k(t)$ replaced respectively by $a_n(t)$, $R_n(t)$ and $k_n(t)$ ($n \in \mathbb{N}_0$). Assume further that there exists a number

$$\omega \geq \sup_{n\in\mathbb{N}_0} \max(0, \operatorname{abs}(|a_n|), \operatorname{abs}(k_n))$$

such that, for every seminorm $p \in \circledast$, there exist a number $c_p > 0$ and a seminorm $r_p \in \circledast$ such that (3.1) holds. Let $\lambda_0 \geq \omega$, and let $\mathfrak{T}$ be defined as above. Assume that the following conditions hold:

(ii) *For every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in E, one has $\sup_{n\in\mathbb{N}} p(C_n x_n) < \infty$.*

(iii) *There exists $\lambda' \in \mathfrak{T}$ such that*

$$R\left(\left(\frac{1}{\tilde{a}(\lambda')} - \mathcal{A}\right)^{-1} C\right)$$

is dense in E, as well as that the sequences

$$(\widetilde{k_n}(\lambda')\widetilde{a_n}(\lambda')^{-1})_{n\in\mathbb{N}} \quad \text{and} \quad (\widetilde{a_n}(\lambda')^{-1})_{n\in\mathbb{N}}$$

are bounded.

(iv) *For every $\epsilon > 0$ and $t \geq 0$, there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that*

$$\int_0^{\min(t,s)} \big|a_n(\max(t,s) - r) - a_n(\min(t,s) - r)\big|\, dr + \int_{\min(t,s)}^{\max(t,s)} \big|a_n(\max(t,s) - r)\big|\, dr < \epsilon,$$

provided $|t-s| < \delta,\ s \geq 0$ and $n \geq n_0$.

Then

$$\lim_{n\to\infty} \widetilde{k_n}(\lambda)\big(I - \widetilde{a_n}(\lambda)\mathcal{A}_n\big)^{-1} C_n x = \tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1} Cx, \quad \lambda \in \mathfrak{T},\ x \in E,$$

is equivalent to say that $\lim_{n\to\infty} R_n(t)x = R(t)x,\ t \geq 0,\ x \in E$, uniformly on compacts of $[0,\infty)$.

Suppose now that $\mathcal{A}$ is a closed MLO, $C\mathcal{A} \subseteq \mathcal{A}C$, $\lambda \in \rho_C(\mathcal{A})$, $R(C)$ and $D(\mathcal{A})$ are dense in E. Then the set $((\lambda - \mathcal{A})^{-1}C)^k(D(\mathcal{A}^n))$ is dense in E for every $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ ([15]). Keeping this in mind, the conclusions from [12, Remark 2.5(ii)] can be formulated in our context; the same holds for the parts (i) and (iii) of this remark. Since subordination principles established in [5] and [20] hold in our framework ([15]), Theorem 3.3 can be used for proving the following extension of [12, Theorem 2.6]:

Theorem 3.4. *Suppose $\alpha > 0$, $\beta \geq 1$, $\mathcal{A}$ is a closed subgenerator of an exponentially equicontinuous (g_α, g_β)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying (2.1) with $a(t) = g_\alpha(t)$ and $k(t) = g_\beta(t)$, and $R(C)$ as well as $D(\mathcal{A})$ are dense in E. Let $(\alpha_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers with $\lim_{n\to\infty} \alpha_n = \alpha$, and let $\gamma_n = \alpha_n/\alpha$ $(n \in \mathbb{N})$. Then, for every $n \in \mathbb{N}$, the operator $\mathcal{A}$ is a subgenerator of an exponentially equicontinuous $(g_{\alpha_n}, g_{1+\gamma_n(\beta-1)})$-regularized C-resolvent family $(R_n(t))_{t\geq 0}$ satisfying (2.1) with $a(t) = g_{\alpha_n}(t)$, $k(t) = g_{1+\gamma_n(\beta-1)}(t)$ and $(R(t))_{t\geq 0}$ replaced by $(R_n(t))_{t\geq 0}$. Furthermore, $\lim_{n\to\infty} R_n(t)x = R(t)x$, $t \geq 0$, $x \in E$, uniformly on compacts of $[0,\infty)$.*

If $\mathcal{A}$ is a subgenerator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$, $l \in \mathbb{N}$ and $x_j \in \mathcal{A}x_{j-1}$ for $1 \leq j \leq l$, then we can prove inductively that, for every $t \in [0,\tau)$,

$$R(t)x = k(t)Cx_0 + \sum_{j=1}^{l-1} \big(a^{*,j} * k\big)(t)Cx_j + \big(a^{*,l} * R(\cdot)x_l\big)(t).$$

Furthermore, for every $x = x_0 \in D(\mathcal{A}^l)$, we can find a sequence $(x_j)_{1\leq j\leq l}$ such that $x_j \in \mathcal{A}x_{j-1}$ for $1 \leq j \leq l$. With this in view, it is very simple to extend the assertion of [12, Theorem 2.7] to (a,k)-regularized C-resolvent families subgenerated by multivalued linear operators.

Theorem 3.5. *Assume that, for every $n \in \mathbb{N}_0$, $|a_n|(t)$ and $k_n(t)$ satisfy* (P1) *and that $\mathcal{A}$ is a closed subgenerator of an (a_n,k_n)-regularized C_n-resolvent family $(R_n(t))_{t\geq 0}$ which satisfies (2.1) with $a(t)$, $R(t)$ and $k(t)$ replaced respectively by $a_n(t)$, $R_n(t)$ and $k_n(t)$ $(n \in \mathbb{N}_0)$. Denote by $a_{n,k}(t)$ the k-th convolution power of the function $a_n(t)$ $(k \in \mathbb{N})$. Assume further that there exists a number $\omega \geq \sup_{n\in\mathbb{N}_0} \max(0, abs(|a_n|), abs(k_n))$ such that, for every seminorm $p \in \circledast$, there exist a number $c_p > 0$ and a seminorm $r_p \in \circledast$ such that (3.1) holds. Let $\lambda_0 \geq \omega$. Suppose that $l \in \mathbb{N}$ and the following holds:*

(i) $\lim_{n\to\infty} \widetilde{k_n}(\lambda)(I - \widetilde{a_n}(\lambda)\mathcal{A})^{-1}C_n x = \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx$ *for $\lambda \in \mathfrak{T}$ and $x \in D(A^l)$.*

(ii) *The sequences* $(k_n(t))_n$, $((a_n * k_n)(t))_n, \ldots,$ *and* $((a_{n,l-1} * k_n)(t))_n$ *are equicontinuous at each point* $t \geq 0$.

(iii) *The sequence* $(C_n x)_n$ *is bounded for any* $x \in D(\mathcal{A}^l)$.

(iv) *The condition* (iv) *of Theorem* 3.3 *holds with the function* $a_n(t)$ *replaced by* $a_{n,l}(t)$.

Then, for every $x \in \overline{D(\mathcal{A}^l)}$, *one has* $\lim_{n\to\infty} R_n(t)x = R(t)x,\ t \geq 0$, *uniformly on compacts of* $[0, \infty)$.

In [15, Theorem 5.12], we have proved the Hille-Yosida theorem for degenerate (a, k)-regularized C-resolvent families. Using this theorem and the argumentation contained in the proof of [12, Theorem 2.8], we can prove the following:

Theorem 3.6. *Assume that, for every* $n \in \mathbb{N}_0$, $|a_n|(t)$ *and* $k_n(t)$ *satisfy* (P1), $\mathcal{A}_n$ *is a closed* MLO, *and*

$$\lambda_0 > \omega \geq \sup_{n \in \mathbb{N}_0} \max(0, \operatorname{abs}(|a_n|), \operatorname{abs}(k_n)).$$

Assume that $\lim_{n\to\infty} \widetilde{a_n}(\lambda) = \tilde{a}(\lambda),\ \lambda \in \mathfrak{T}$ *and* $\lim_{n\to\infty} \widetilde{k_n}(\lambda) = \tilde{k}(\lambda),\ \lambda \in \mathfrak{T}$. *Suppose that* $L(E) \ni \tilde{k}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C,\ \lambda \in \mathfrak{T}$, *and for every* $n \in \mathbb{N}$, $\mathcal{A}_n$ *is a subgenerator of an* (a_n, k_n)*-regularized* C_n*-resolvent family* $(R_n(t))_{t\geq 0}$ *which satisfies* (2.1) *with* $a(t)$, $R(t)$ *and* $k(t)$ *replaced respectively by* $a_n(t)$, $R_n(t)$ *and* $k_n(t)$. *Let* (3.1) *hold for* $t \geq 0$, $x \in E$ *and* $n \in \mathbb{N}$, *and let*

$$\lim_{n\to\infty} \widetilde{k_n}(\lambda)\big(I - \widetilde{a_n}(\lambda)\mathcal{A}_n\big)^{-1} C_n x = \tilde{k}(\lambda)\big(I - \tilde{a}(\lambda)\mathcal{A}\big)^{-1} Cx, \quad x \in E,\ \lambda \in \mathfrak{T}.$$

Suppose, further, that for each $\lambda \in \mathfrak{T}$ *there exists an open ball*

$$\Omega_\lambda \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > \lambda_0\},$$

with center at point λ *and radius* $2\epsilon_\lambda > 0$, *so that* $\widetilde{a_n}(z)\widetilde{k_n}(z) \neq 0$, $z \in \Omega_\lambda$, $n \in \mathbb{N}_0$. *Then the following holds:*

(i) *For each* $r \in (0, 1]$, $\mathcal{A}$ *is a subgenerator of a global* $(a, k * g_r)$*-regularized* C*-resolvent family* $(R_r(t))_{t\geq 0}$ *satisfying* (2.1) *as well as that, for every seminorm* $p \in \circledast$,

$$p\big(R_r(t+h)x - R_r(t)x\big) \leq \frac{2c_p r_p(x) h^r}{r\Gamma(r)} \max\big(e^{\omega(t+h)}, 1\big),\ t \geq 0,\ h > 0,\ x \in E,$$

and that, for every seminorm $p \in \circledast$ *and bounded set* $B \in \mathcal{B}$, *the mapping* $t \mapsto p_B(R_r(t))$, $t \geq 0$, *is locally Hölder continuous with exponent* r.

(ii) *If $\mathcal{A}$ is densely defined, then $\mathcal{A}$ is a subgenerator of a global (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying* (2.1) *and that the family* $\{\mathrm{e}^{-\omega t}R(t) : t \geq 0\}$ *is equicontinuous.*

Suppose that $\mathcal{A}$ is an MLO, $C\mathcal{A} \subseteq \mathcal{A}C$ and $\rho_C(\mathcal{A}) \neq \emptyset$. Then, for every $\lambda \in \rho_C(\mathcal{A})$, we have $\mathcal{A}0 = N((\lambda I - \mathcal{A})^{-1}C)$, which implies that the operator $(\lambda I - \mathcal{A})^{-1}C$ is injective iff $\mathcal{A}$ is single-valued. Although the resolvent equation

$$\big(\lambda - \mathcal{A}\big)^{-1}C^2x - \big(\mu - \mathcal{A}\big)^{-1}C^2x = (\mu - \lambda)\big(\lambda - \mathcal{A}\big)^{-1}C\big(\mu - \mathcal{A}\big)^{-1}Cx,\ x \in E$$

holds for any $\lambda,\ \mu \in \rho_C(\mathcal{A})$, the non-injectivity of operator $(\lambda I - \mathcal{A})^{-1}C$ in multivalued case does not permit us to state a satisfactory extension of [12, Corollary 2.10] for degenerate resolvent families. The assertion of [12, Proposition 2.11(i)] can be formulated in our context (cf. Lemma 2.1), which is not the case with the assertions of [12, Proposition 2.11(ii)] and [12, Proposition 2.12].

The author has recently analyzed abstract degenerate Volterra equations of non-scalar type in [14]. The interested reader may try to prove some results on approximation and convergence of degenerate (A,k)-regularized C-(pseudo)resolvent families introduced in this paper (cf. [12, Theorem 2.16, Theorem 2.17] for non-degenerate case), as well as degenerate (a,k)-regularized C-resolvent families introduced in the papers [16]–[17].

4. *Laguerre expansions of degenerate (a,k)-regularized C-resolvent families*

In this section, we shall present the basic results about Laguerre expansions of degenerate (a,k)-regularized C-resolvent families in locally convex spaces (cf. the recent paper by L. Abadias and P. J. Miana [1] for C_0-semigroup case).

We start by recalling that Rodrigues' formula gives the following representation of generalized Laguerre polynomials

$$L_n^{\alpha}(t) \equiv \mathrm{e}^t \frac{t^{-\alpha}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(\mathrm{e}^{-t}t^{n+\alpha}\big), \quad t \in \mathbb{R} \quad \big(n \in \mathbb{N}_0,\ \alpha > -1\big).$$

If $\alpha \notin -\mathbb{N}$ and $n \in \mathbb{N}_0$, then we define

$$l_n^{\alpha}(t) \equiv \frac{1}{\Gamma(n+\alpha+1)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\big(\mathrm{e}^{-t}t^{n+\alpha}\big), \quad t > 0.$$

The reader may consult [1, Section 2] for the most important properties of functions $l_n^{\alpha}(t)$ ($\alpha \notin -\mathbb{N}$, $n \in \mathbb{N}_0$). For example, it is well known that

$$l_n^{\alpha}(t) \sim g_{\alpha+1}(t),\ t \to 0+\ ;\ \ l_n^{\alpha}(t) \sim (-1)^n e^{-t} g_{n+\alpha+1}(t),\ t \to +\infty, \tag{4.1}$$

and

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} l_n^{\alpha}(t) = l_{n+k}^{\alpha-k}(t), \quad t > 0, \ k \in \mathbb{N}_0. \tag{4.2}$$

The following theorem can be deduced by using the argumentation contained in the proof of [1, Theorem 3.3], stated in the Banach space case (cf. [19, Theorem 3, Section 4.23] for scalar-valued case).

Theorem 4.1. *Suppose that $f : (0, \infty) \to E$ is a differentiable mapping, $\alpha > -1$ and for each seminorm $p \in ⊛$ we have $\int_0^\infty \mathrm{e}^{-t} t^\alpha p(f(t))^2 \, \mathrm{d}t < \infty$. Then*

$$f(t) = \sum_{n=0}^{\infty} \frac{n! L_n^\alpha(t)}{\Gamma(n+\alpha+1)} \int_0^\infty \mathrm{e}^{-s} s^\alpha L_n^\alpha(s) f(s) \, \mathrm{d}s, \quad t > 0.$$

Since

$$\int_0^\infty \mathrm{e}^{-s} s^\alpha L_n^\alpha(s) f(s) \, \mathrm{d}s = \int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}s^n} \left(\mathrm{e}^{-s} s^{n+\alpha}\right) \frac{f(s)}{n!} \, \mathrm{d}s$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (n+\alpha) \cdots (n+\alpha-(k-1)) \int_0^\infty \mathrm{e}^{-s} s^{n+\alpha-k} \frac{f(s)}{n!} \, \mathrm{d}s$$

and

$$\frac{(n+\alpha)\cdots(n+\alpha-(k-1))}{\Gamma(n+\alpha+1)} = \frac{1}{\Gamma(n+\alpha+1-k)},$$

for any $n, \ k \in \mathbb{N}_0$ with $k \leq n$ and $\alpha > -1$, we immediately obtain the following corollary of Theorem 4.1.

Corollary 4.1. *Suppose that $f : (0, \infty) \to E$ is a differentiable mapping, $\alpha > -1$ and for each seminorm $p \in ⊛$ we have $\int_0^\infty \mathrm{e}^{-s} s^\alpha p(f(s))^2 \, \mathrm{d}s < \infty$. Then, for every $t > 0$, the following equality holds:*

$$f(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{L_n^\alpha(t)(-1)^{n-k}\binom{n}{k}}{\Gamma(n+\alpha+1-k)} \int_0^\infty \mathrm{e}^{-s} s^{n+\alpha-k} f(s) \, \mathrm{d}s. \tag{4.3}$$

Before proceeding further, let us observe that the last formula can be rewritten in the following equivalent way:

$$f(t) = \sum_{n=0}^{\infty} L_n^\alpha(t) \int_0^\infty l_n^\alpha(s) f(s) \, \mathrm{d}s.$$

Suppose now that $(R(t))_{t\geq 0}$ is an exponentially equicontinuous (a,k)-regularized C-resolvent family with a closed subgenerator $-\mathcal{A}$, the functions $k(t)$ and $|a|(t)$ satisfy (P1), the family $\{\mathrm{e}^{-\omega t}R(t) : t \geq 0\}$ is equicontinuous for some $\omega \geq 0$, $\omega_0 \equiv \max(\omega, \mathrm{abs}(|a|), \mathrm{abs}(k)) < 1/2$ and $\alpha > -1$. If, in addition, $\tilde{k}(1)\tilde{a}(1) \neq 0$, then Theorem 2.3 implies that, for every $\alpha \in \mathbb{N}_0$ and $x \in E$,

$$\begin{aligned}\int\limits_0^\infty & \mathrm{e}^{-s} s^{n+\alpha-k} R(s)x\,\mathrm{d}s \\ &= (-1)^{n+\alpha-k}\left(\frac{\mathrm{d}^{n+\alpha-k}}{\mathrm{d}\lambda^{n+\alpha-k}}\big(\mathcal{L}R(\cdot)x\big)(\lambda)\right)_{\lambda=1} \\ &= (-1)^{n+\alpha-k}\left(\frac{\mathrm{d}^{n+\alpha-k}}{\mathrm{d}\lambda^{n+\alpha-k}}\left[\frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\Big(\frac{1}{\tilde{a}(\lambda)}+\mathcal{A}\Big)^{-1}Cx\right]\right)_{\lambda=1}; \qquad (4.4)\end{aligned}$$

then one can use the product rule, the following identity from [15]

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\big(\lambda+\mathcal{A}\big)^{-1}Cx = (-1)^n n!\big(\lambda+\mathcal{A}\big)^{-n-1}Cx, \quad n \in \mathbb{N}_0,\ x \in E,$$

the equation (4.4), as well as the well known Faà di Bruno's formula

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(g(x)) = \sum \frac{n!}{m_1!m_2!\cdots m_n!} f^{(m_1+m_2+\cdots+m_n)}(g(x)) \prod_{j=1}^n \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j},$$

where the summation is taken over those multi-indices $(m_1, m_2, \ldots, m_n) \in \mathbb{N}_0^n$ for which $m_1 + 2m_2 + \cdots + nm_n = n$, in order to express the right hand side of (4.3), with $f(t) = R(t)x$, $t > 0$, in terms of subgenerator $-\mathcal{A}$ (notice, however, that it is very difficult to express the value of $\int_0^\infty \mathrm{e}^{-s}s^{n+\alpha-k}R(s)x\,\mathrm{d}s$ in terms of $-\mathcal{A}$, if $\alpha \notin \mathbb{N}_0$ and $n \in \mathbb{N}_0$). At any rate, the obtained representation formula is very complicated, practically almost irrelevant, but can be simplified in some cases; for example, if $k(t) = 1$ and $a(t) = g_\vartheta(t)$ for some $\vartheta > 0$, then we have

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}&\left[\frac{\tilde{k}(\lambda)}{\tilde{a}(\lambda)}\Big(\frac{1}{\tilde{a}(\lambda)}+\mathcal{A}\Big)^{-1}Cx\right] = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[\lambda^{\vartheta-1}\big(\lambda^\vartheta+\mathcal{A}\big)^{-1}Cx\right] \\ &= (-1)^n\lambda^{-(n+1)}\sum_{k=1}^{n+1} b_{k,n+1}^\vartheta \lambda^{\vartheta k}\big(\lambda^\vartheta+\mathcal{A}\big)^{-k}Cx, \quad n \in \mathbb{N}_0,\ x \in E,\ \mathrm{Re}\,\lambda > \omega_0,\end{aligned}$$

where the numbers $b^{\vartheta}_{k,n+1}$ are given by the following recurrence relations:

$$b^{\vartheta}_{1,1} = 1,$$

$$b^{\vartheta}_{k,n} = (n-1-k\vartheta)b^{\vartheta}_{k,n-1} + \vartheta b^{\vartheta}_{k-1,n-1}, \quad 1 \le k \le n,\ n = 2,3,\ldots,$$

$$b^{\vartheta}_{k,n} = 0, \quad k > n,\ n = 1,2,\ldots,$$

cf. the formulae [5, (2.16)–(2.17)].

Laguerre expansions can be elegantly used for proving some representation formulae for solutions of abstract non-degenerate differential equations of first order whose solutions are goverened by fractionally integrated semigroups and exponential ultradistribution semigroups of Beurling class. Furthermore, we can consider Laguerre expansions of certain classes of semigroups that are strongly continuous for $t > 0$, like semigroups of class $(C_{(k)})$ and semigroups of growth order $r > 0$. For more details about the above-mentioned topics, we refer the reader to [11].

In [2], L. Abadias and P. J. Miana have recently analyzed the Hermite expansions of non-degenerate C_0-groups and cosine operator functions in Banach spaces. The interested reader may try to reconsider the results from [2] for some other classes of (non-)degenerate resolvent operator families.

REFERENCES

[1] L. Abadias, P. J. Miana, *C_0-semigroups and resolvent operators approximated by Laguerre expansions,* J. Approx. Theory **213** (2017), 1–22.

[2] L. Abadias, P. J. Miana, *Hermite expansions of C_0-groups and cosine families,* J. Math. Anal. Appl. **426** (2015), 288–311.

[3] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser Verlag, 2001.

[4] W. Arendt, *Approximation of degenerate semigroups,* Taiwanese J. Math. **5** (2001), 279–295.

[5] E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces,* Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, 2001.

[6] V. Barbu, A. Favini, *Convergence of solutions of implicit differential equations,* Diff. Int. Eqs. **7** (1994), 665–688.

[7] R. W. Carroll, R. W. Showalter, *Singular and Degenerate Cauchy Problems,* Academic Press, New York, 1976.

[8] R. Cross, *Multivalued Linear Operators,* Marcel Dekker Inc., New York, 1998.

[9] A. Favini, A. Yagi, *Degenerate Differential Equations in Banach Spaces,* Chapman and Hall/CRC Pure and Applied Mathematics, New York, 1998.

[10] M. Kostić, *Abstract Volterra Integro-Differential Equations*, CRC Press, Boca Raton, Fl, 2015.

[11] M. Kostić, *Abstract Degenerate Volterra Integro-Differential Equations: Linear Theory and Applications,* Book Manuscript, 2016.

[12] M. Kostić, *Approximations and convergence of (a, k)-regularized C-resolvent families,* Numer. Funct. Anal. Appl. **35** (2014), 1579–1606.

[13] M. Kostić, *Abstract Volterra equations in locally convex spaces*, Sci. China Math. **55** (2012), 1797–1825.

[14] M. Kostić, *Abstract degenerate non-scalar Volterra equation,* Chelyabinsk Phy. Math. J. **1** (2016), 99–106.

[15] M. Kostić, *Abstract degenerate Volterra inclusions in locally convex spaces*, preprint.

[16] M. Kostić, *Degenerate abstract Volterra equations in locally convex spaces,* Filomat **31** (2017), 597–619.

[17] M. Kostić, *Degenerate multi-term fractional differential equations in locally convex spaces,* Publ. Inst. Math., Nouv. Sér. **100** (2016), 49–75.

[18] R. K. Lamm, I. G. Rosen, *An approximation theory for the estimation of parameters in degenerate Cauchy problems,* J. Math. Anal. Appl. **162** (1991), 13–48.

[19] N. N. Lebedev, *Special Functions and Their Applications,* Selected Russian Publications in the Mathematical Sciences. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965.

[20] J. Prüss, *Evolutionary Integral Equations and Applications,* Birkhäuser-Verlag, Basel, 1993.

[21] G. A. Sviridyuk, V. E. Fedorov, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators,* Inverse and Ill-Posed Problems (Book **42**), VSP, Utrecht, Boston, 2003.

[22] B. Thaller, S. Thaller, *Approximation of degenerate Cauchy problems,* SFB F0003, Optimierung und Kontrolle 76, University of Graz, 13 pp.

Faculty of Technical Sciences
University of Novi Sad
Trg Dositeja Obradovića 6
Novi Sad 21125, Serbia
e-mail: marco.s@verat.net